How to Prove Hybrid Systems

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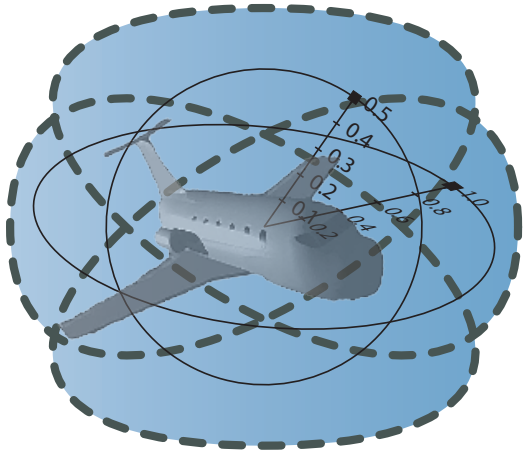


Outline

Cyber-Physical Systems Analysis: Aircraft Example

Which control decisions are safe for aircraft collision avoidance?

Cyber-Physical Systems

CPSs combine cyber capabilities with physical capabilities to solve problems that neither part could solve alone.

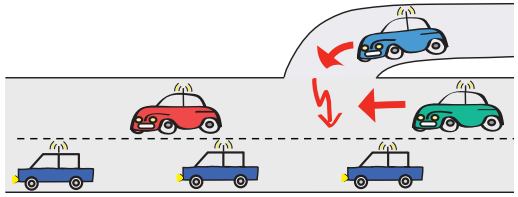
CPSs Promise Transformative Impact!

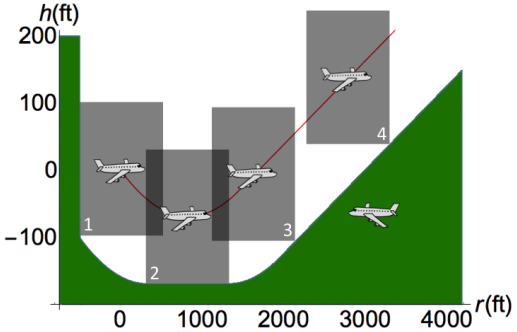
Prospects: Safe & Efficient

Driver assistance
Autonomous cars

Pilot decision support
Autopilots / UAVs

Train protection
Robots near humans





Prerequisite: CPSs need to be safe

How do we make sure CPSs make the world a better place?

Can you trust a computer to control physics?

Can you trust a computer to control physics?

1. Depends on how it has been programmed
2. And on what will happen if it malfunctions

Rationale

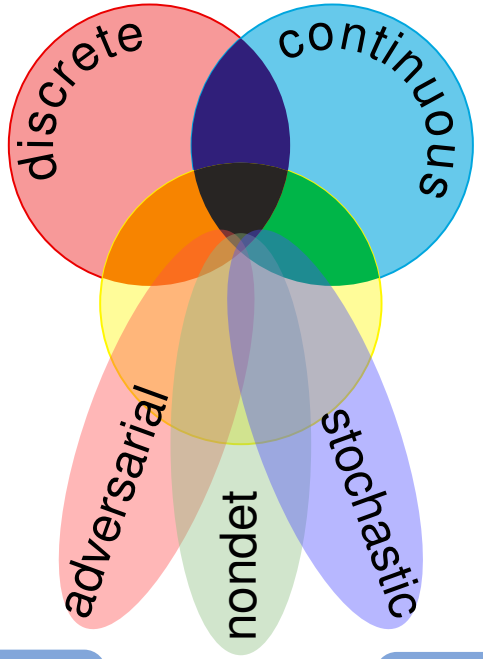
1. Safety guarantees require analytic foundations.
2. A common foundational core helps all application domains.
3. Foundations revolutionized digital computer science & our society.
4. Need even stronger foundations when software reaches out into our physical world.

CPSs deserve proofs as safety evidence!

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

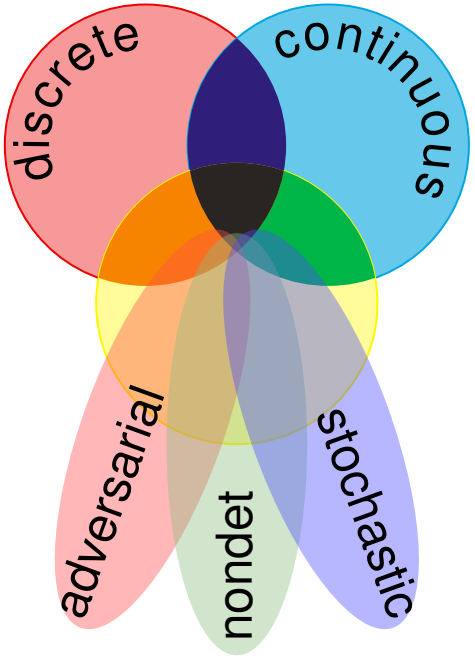
Exploiting compositionality tames CPS complexity.

Analytic simplification

CPSs are Multi-Dynamical Systems

hybrid systems

HS = discrete + ODE

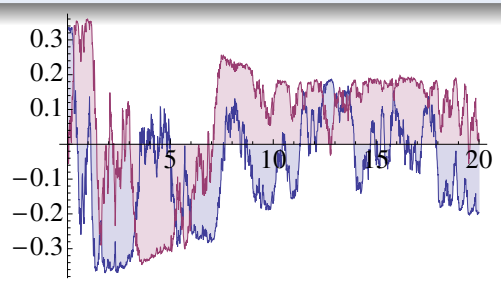


hybrid games

HG = HS + adversary

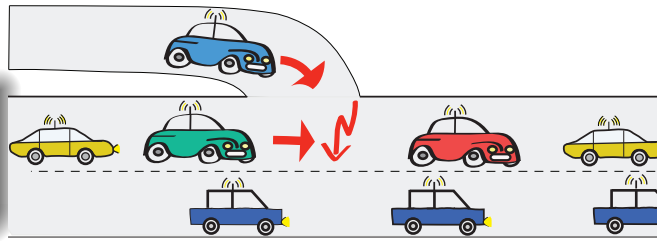
stochastic hybrid sys.

SHS = HS + stochastics



distributed hybrid sys.

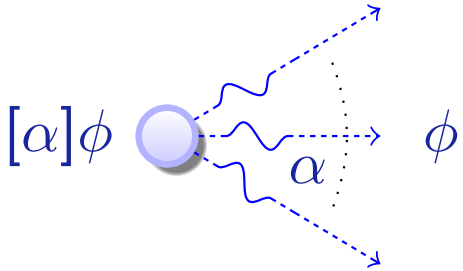
DHS = HS + distributed

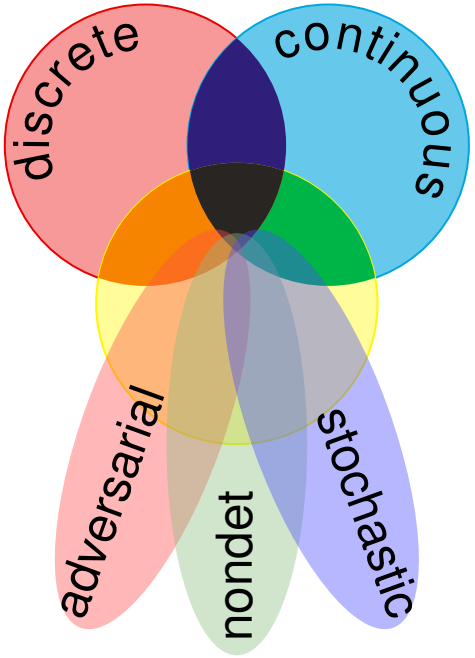


Dynamic Logics for Dynamical Systems

differential dynamic logic

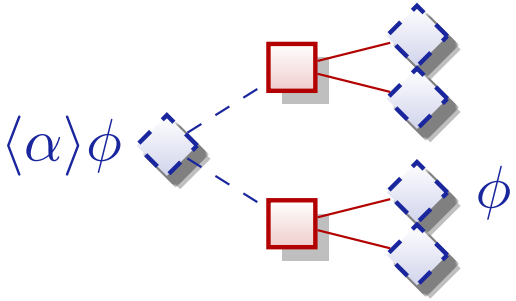
$\mathsf{d}\mathcal{L} = \mathsf{DL} + \mathsf{HP}$





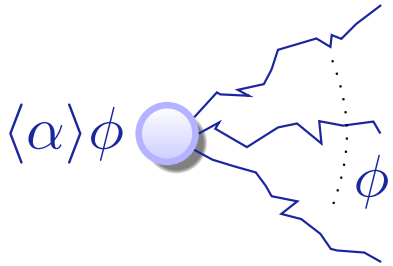
differential game logic

$\mathsf{dG}\mathcal{L} = \mathsf{GL} + \mathsf{HG}$



stochastic differential DL

$\mathsf{Sd}\mathcal{L} = \mathsf{DL} + \mathsf{SHP}$



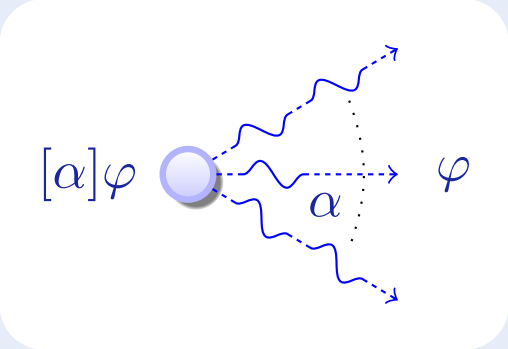
quantified differential DL

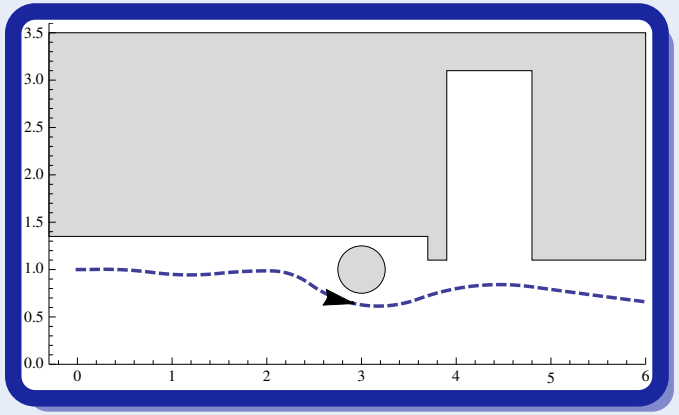
$\mathsf{Qd}\mathcal{L} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$

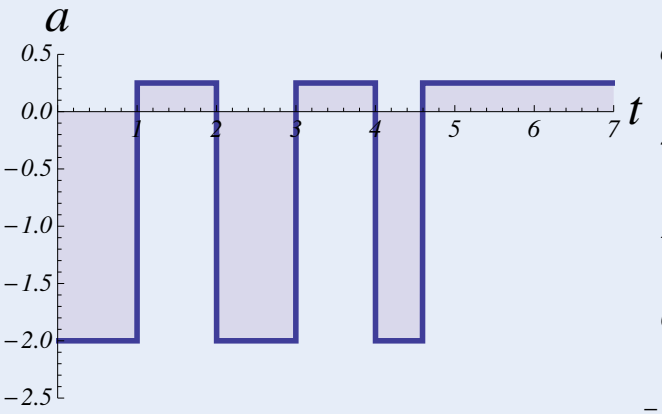
JAR'08,CADE'11,LMCS'12,LICS'12

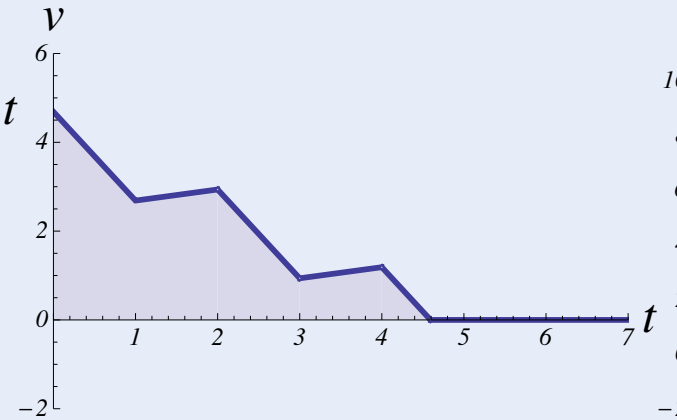
LICS'12,CADE'15,TOCL'15

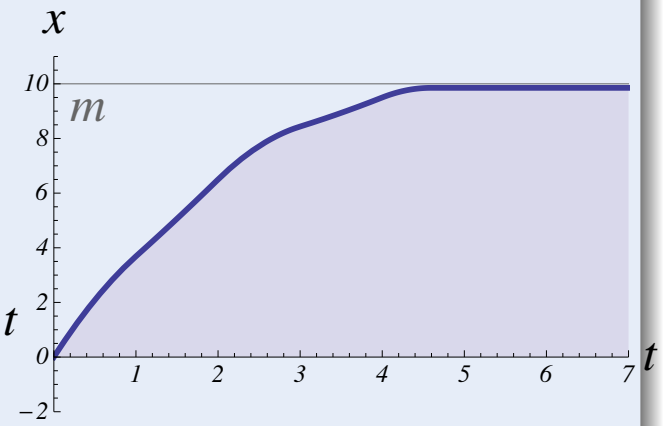
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)











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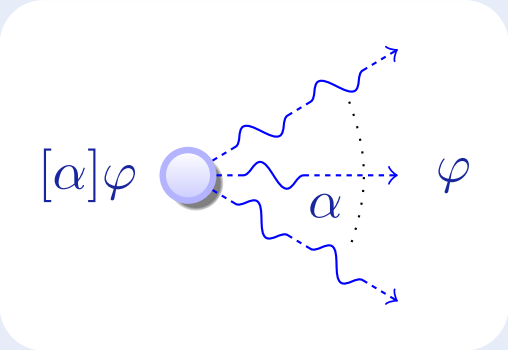

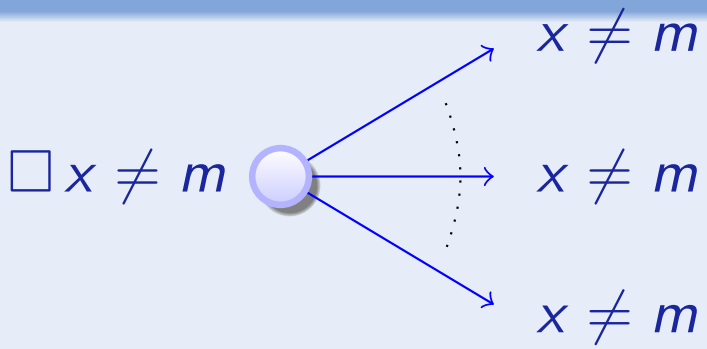

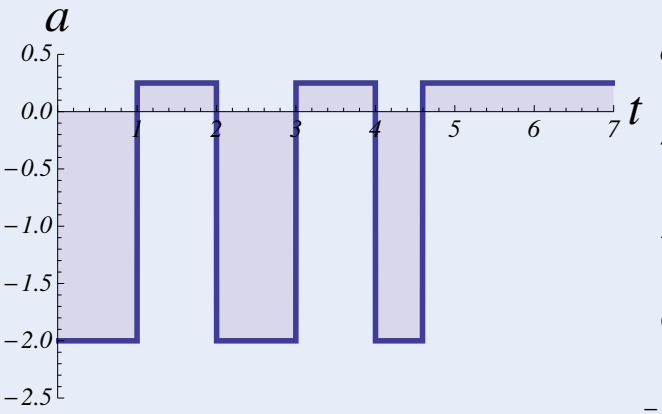

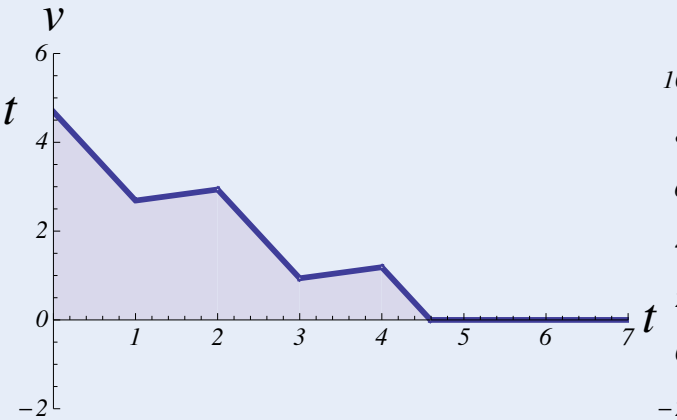

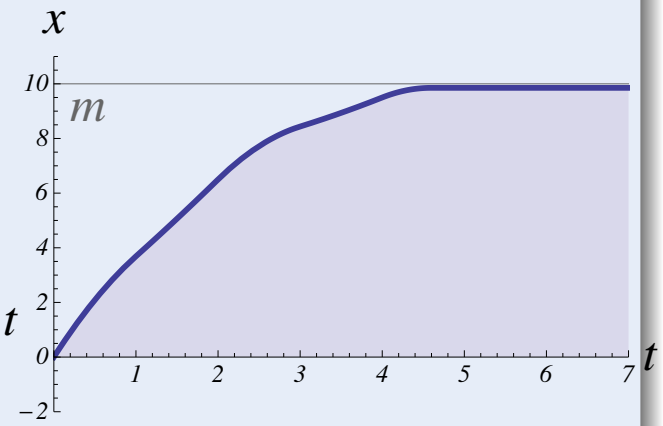

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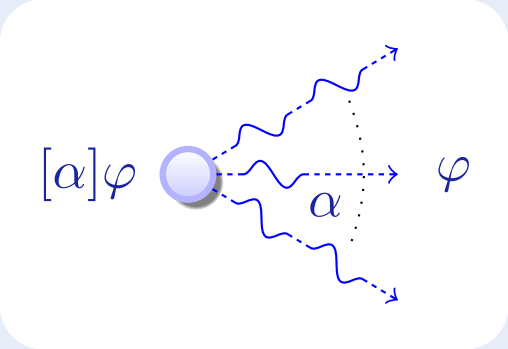

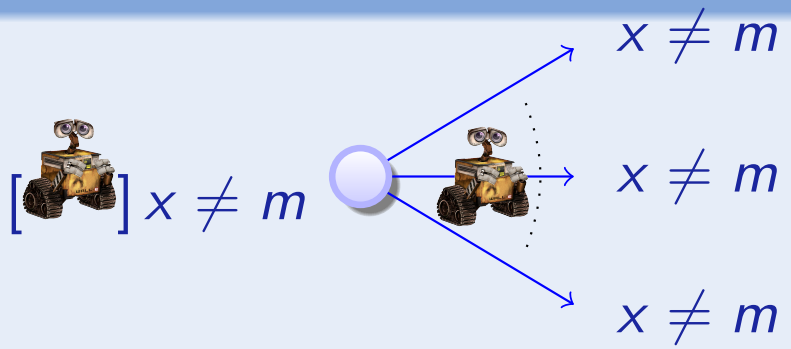

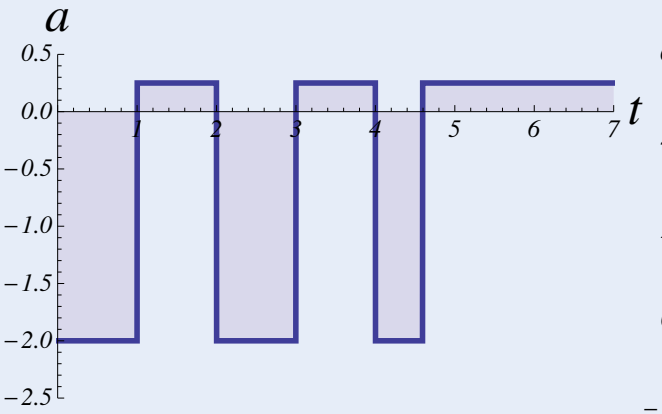

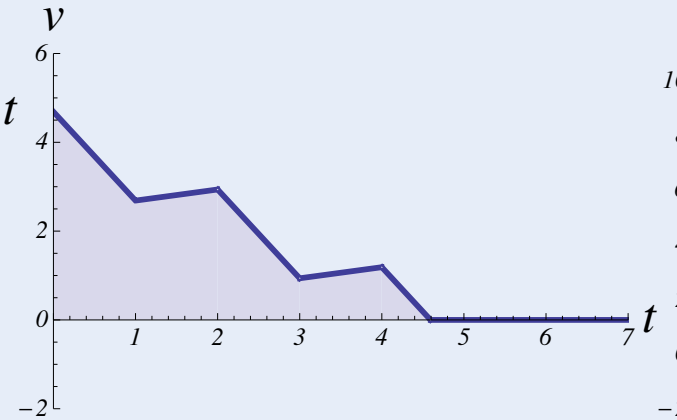

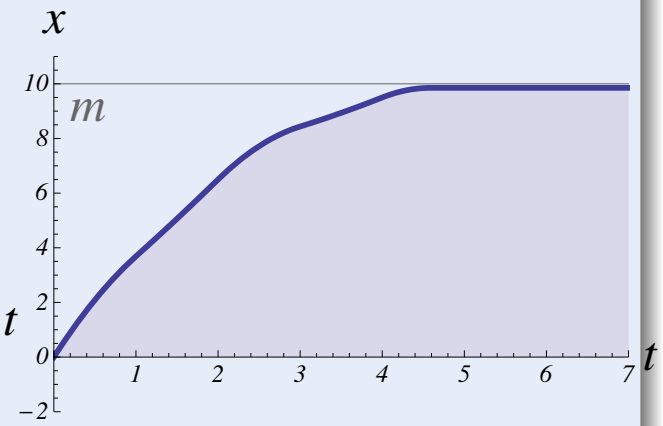

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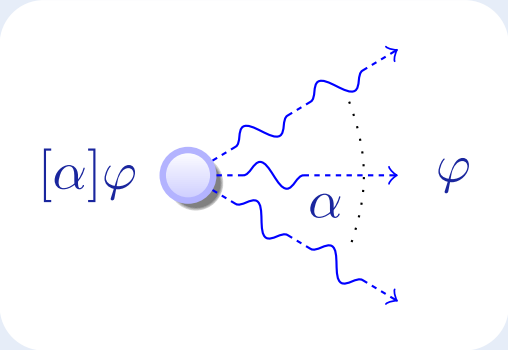
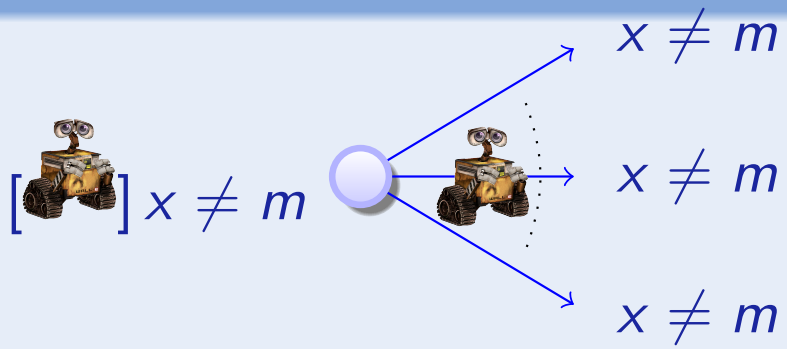
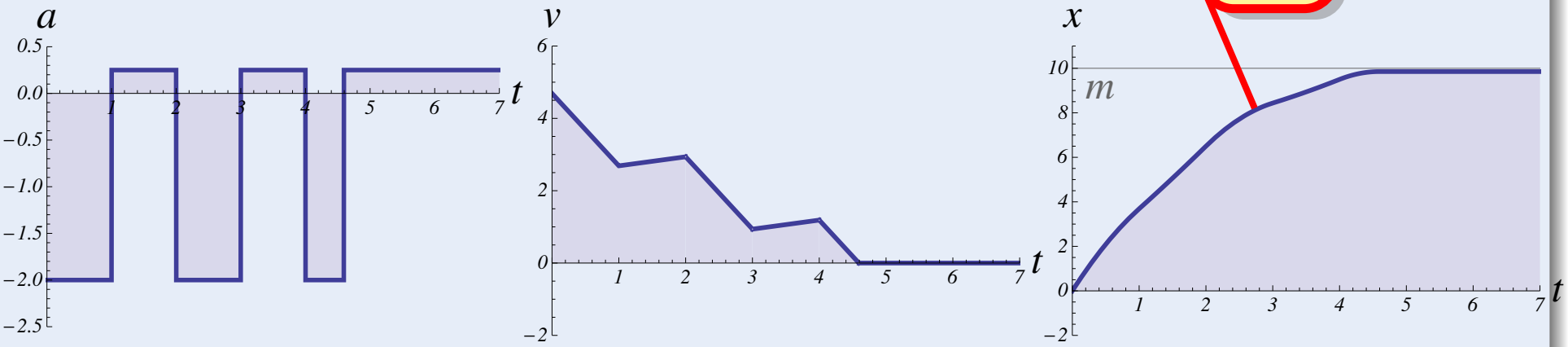




$x' = v, v' = a$

ODE



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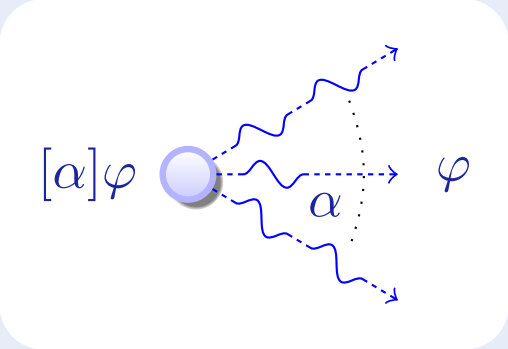
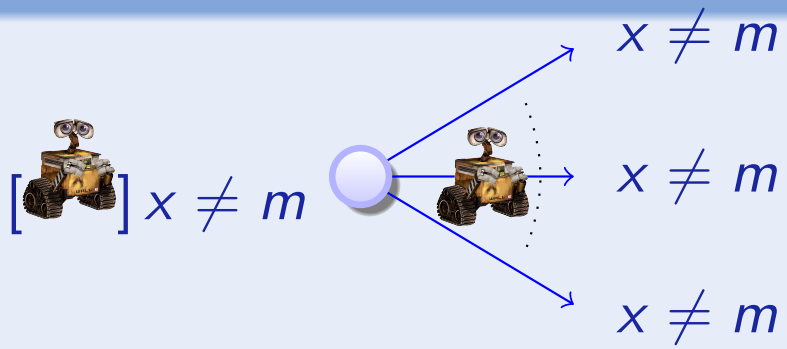
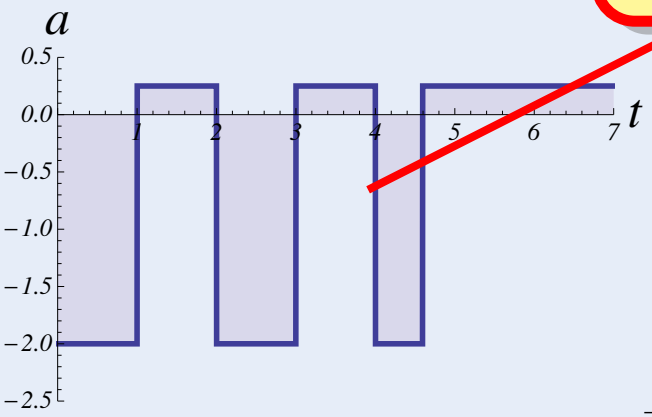


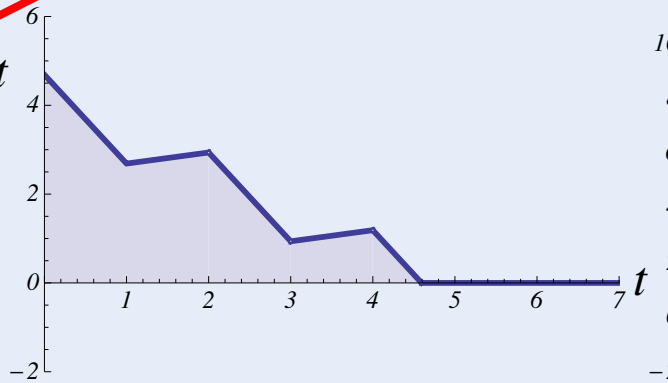


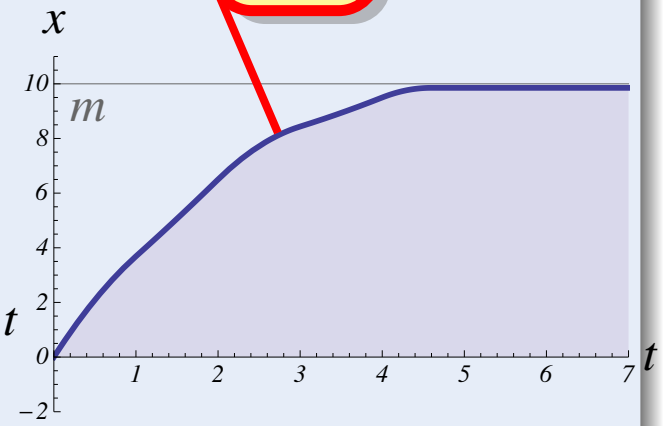
$a := -b$ $\quad x' = v, v' = a$

assign

ODE







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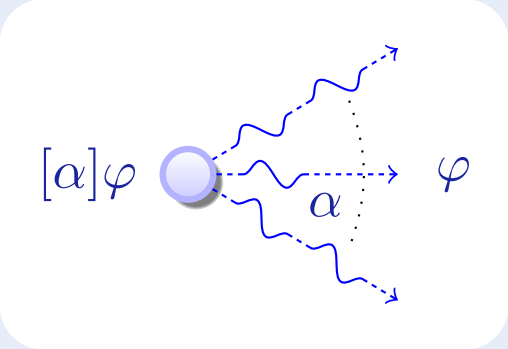
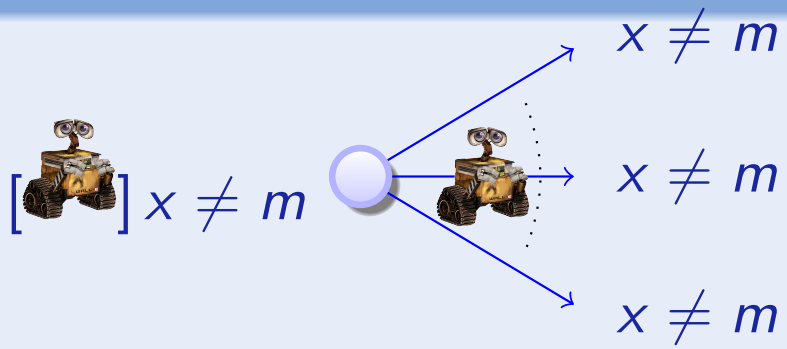
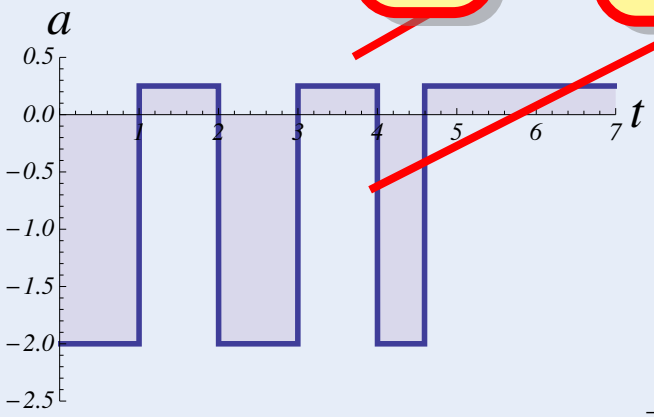
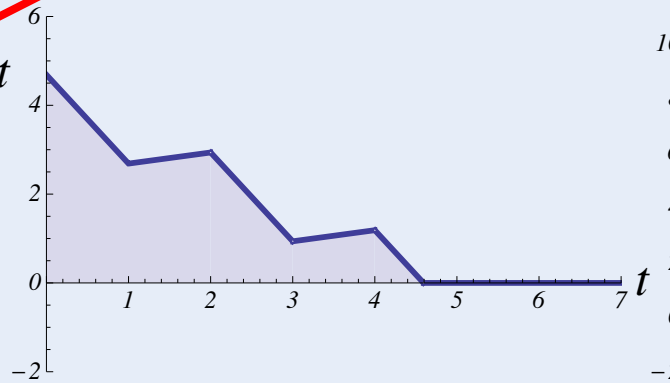


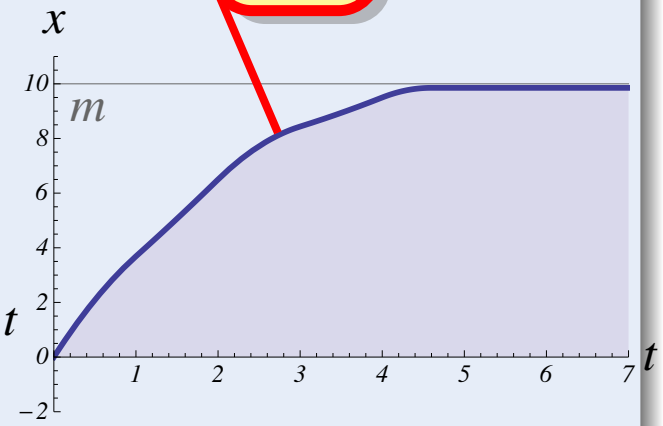


$(\texttt{if}(\mathrm{SB}(x,m))\ a := -b)$ $\quad x' = v, v' = a$

test assign ODE







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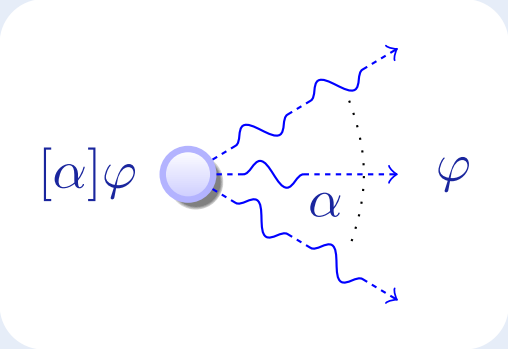


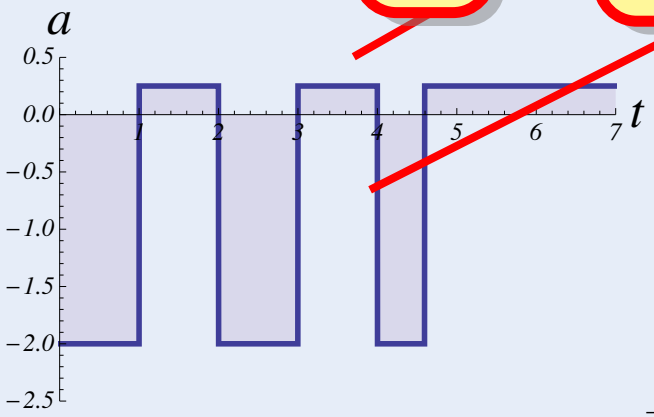
seq. compose

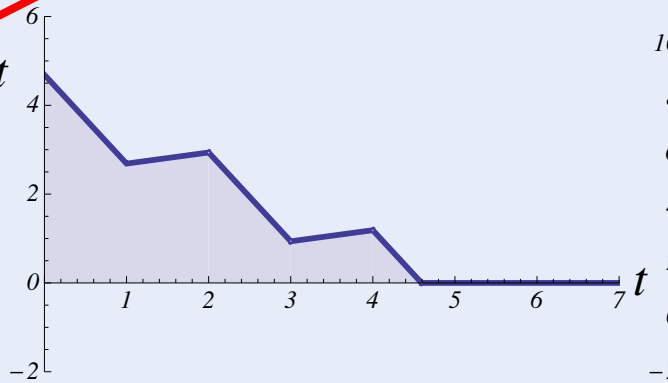
$$(\texttt{if}(\mathsf{SB}(x,m))\ a := -b)\ ;\ x' = v, v' = a$$

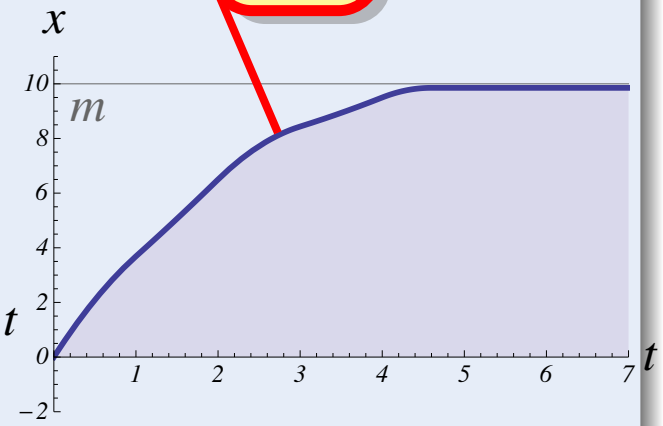
test

assign

ODE







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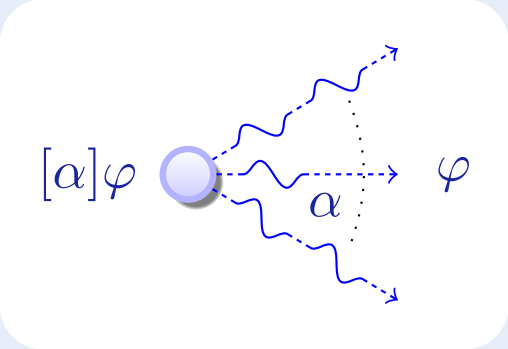


seq. compose

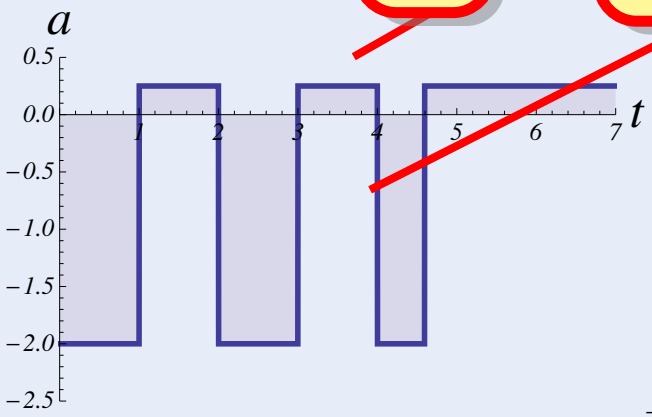
nondet. repeat

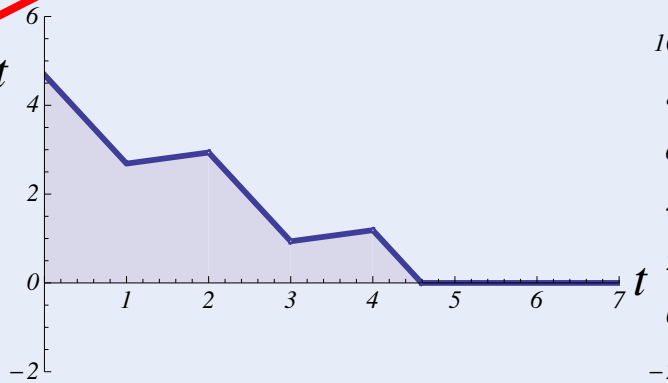
$$\big((\texttt{if}(\mathsf{SB}(x,m))\ a := -b)\ ;\ x' = v, v' = a\big)^*$$

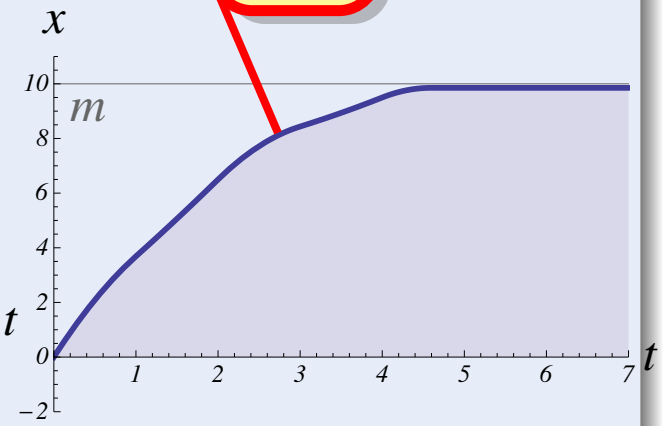
test

assign

ODE







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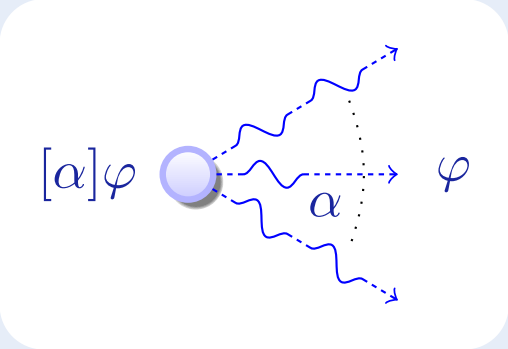
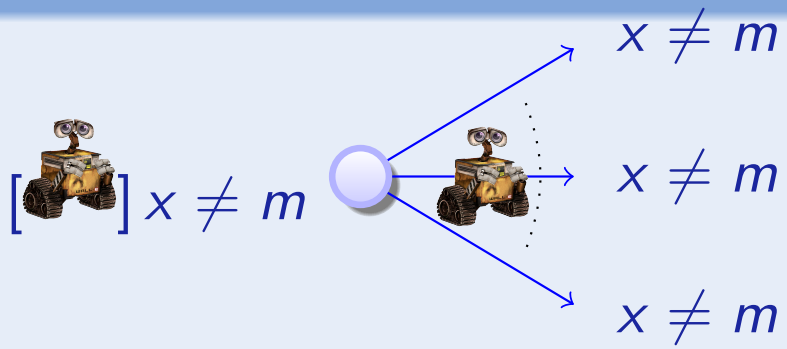
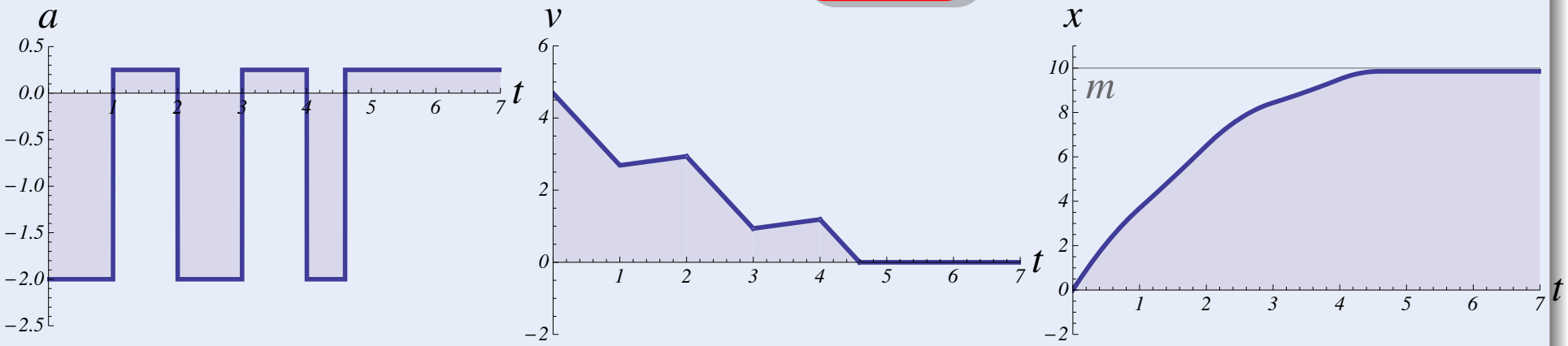




$$\underbrace{[((\texttt{if}(\mathrm{SB}(x,m))\ a := -b)\ ;\ x' = v, v' = a)^*]}_{\text{all runs}}\underbrace{x \neq m}_{\text{post}}$$



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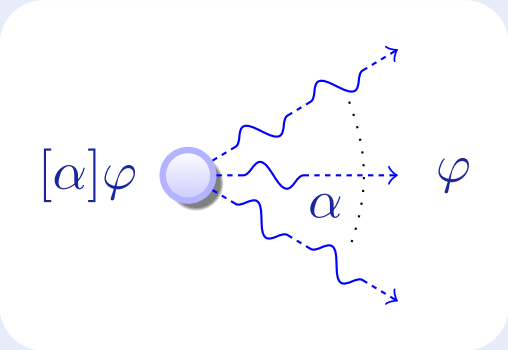
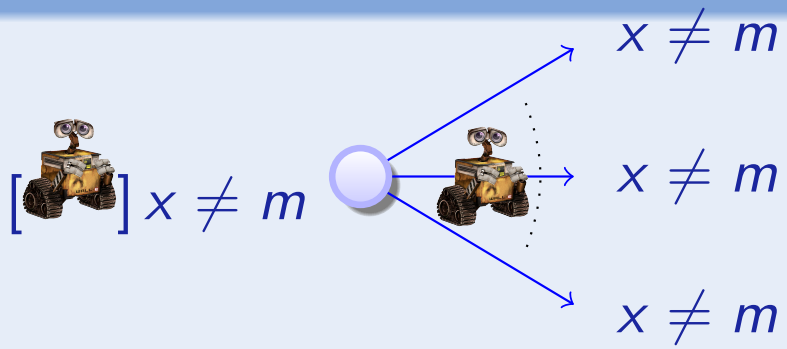
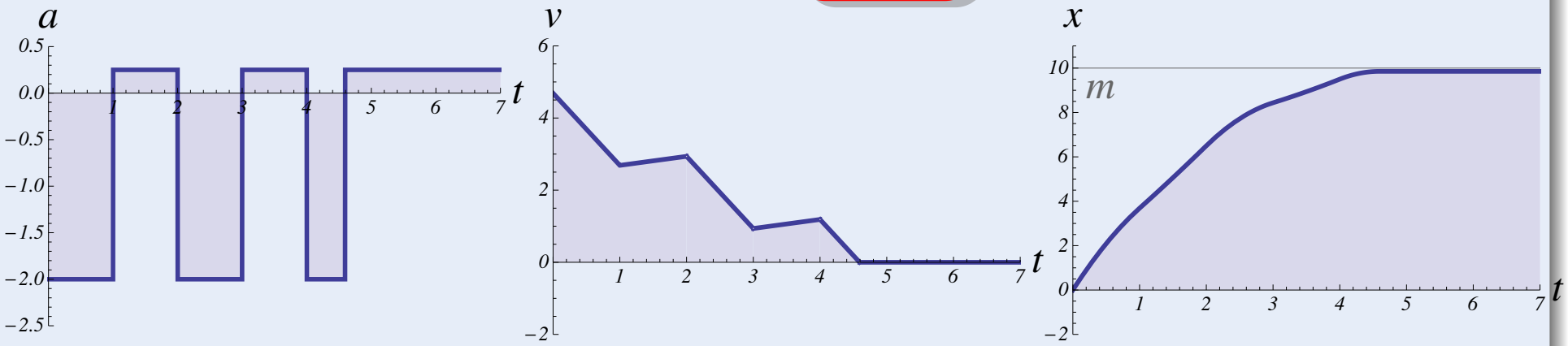




$$\underbrace{x \neq m \wedge b > 0}_{\text{init}} \rightarrow \underbrace{[((\texttt{if}(\mathrm{SB}(x,m))\ a := -b)\ ;\ x' = v, v' = a)^*]}_{\text{all runs}}\underbrace{x \neq m}_{\text{post}}$$



Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := f(x) \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Tableaux'07,JAutomReas'08,LICS'12

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := f(x) \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

All Reals | Some Reals | All Runs | Some Runs

Tableaux'07,JAutomReas'08,LICS'12

Differential Dynamic Logic: Axiomatization

equations of truth

$[:=]$ $[x := e]P(x) \leftrightarrow P(e)$

$[?]$ $[?Q]P \leftrightarrow (Q \rightarrow P)$

$[']$ $[x' = f(x)]P \leftrightarrow \forall t{\geq}0\, [x := y(t)]P$ $\qquad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

I $[\alpha^*](P \rightarrow [\alpha]P) \rightarrow (P \rightarrow [\alpha^*]P)$

C $[\alpha^*]\forall v{>}0\, (P(v) \rightarrow \langle\alpha\rangle P(v{-}1)) \rightarrow \forall v\, (P(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\, P(v))$

LICS'12,JAR'16

Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (J.Autom.Reas. 2008, LICS'12)

$d\mathcal{L}$ calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations **or** *to discrete dynamics.* ▸ Proof 25pp

Corollary (Complete Proof-theoretical Bridge)

proving continuous = proving hybrid = proving discrete

JAutomReas'08,LICS'12

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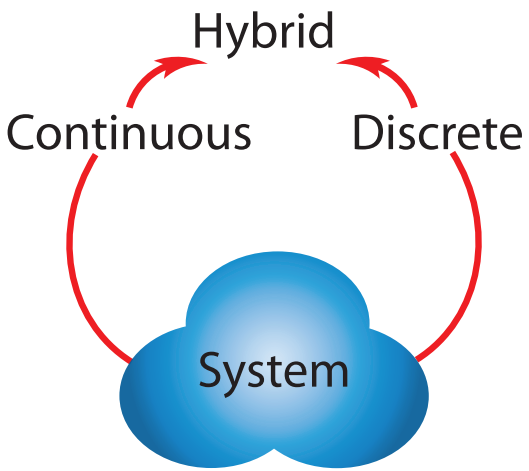

JAutomReas'08,LICS'12

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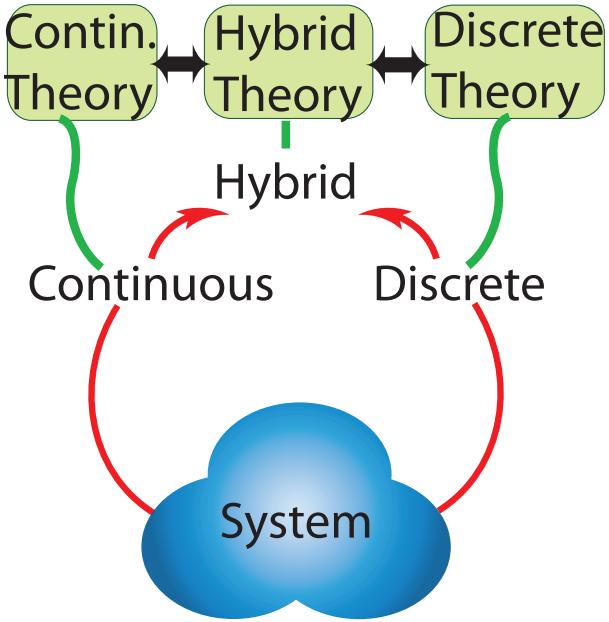
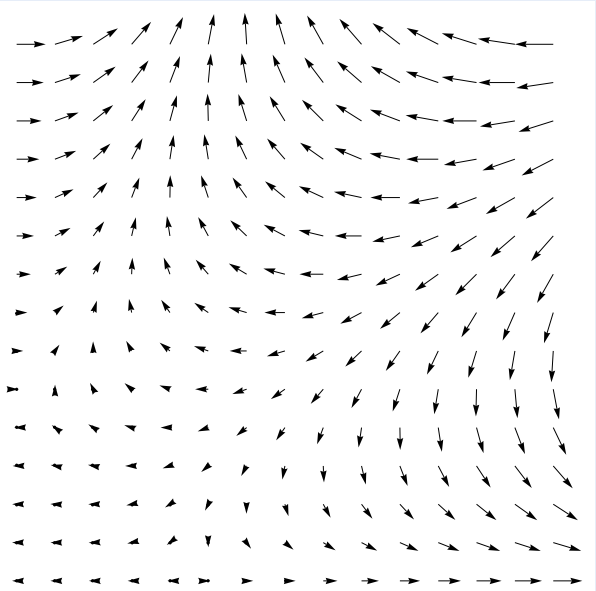


JAutomReas'08,LICS'12

Differential Invariants for Differential Equations

Differential Invariant



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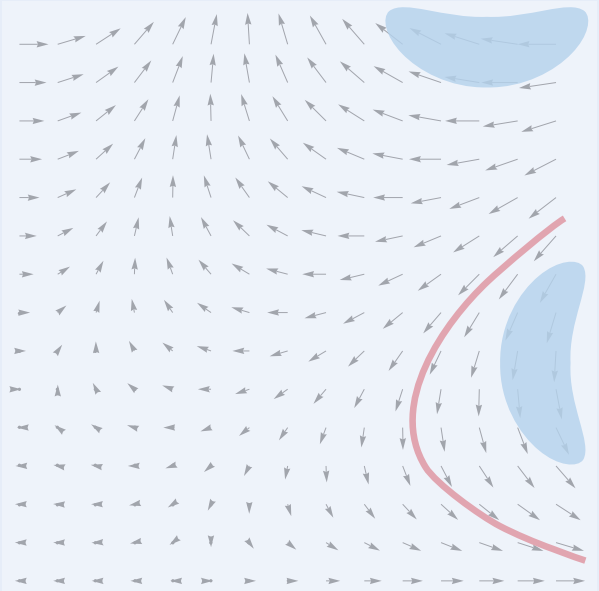

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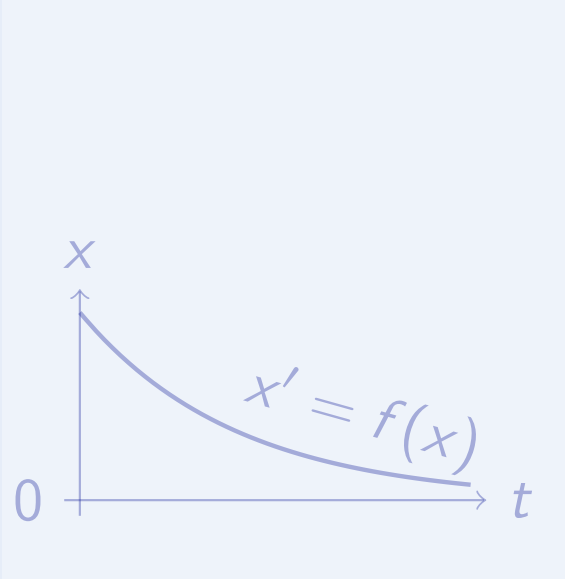
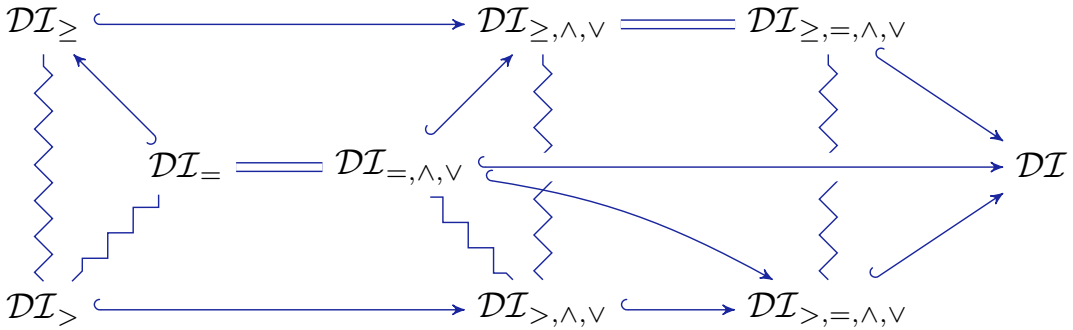




Logic

Provability theory

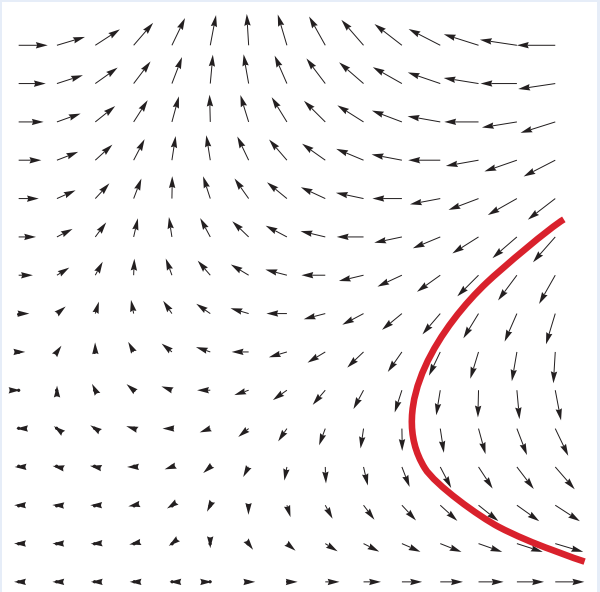
Math

Characteristic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,LICS'12,ITP'12,JAR'16

Differential Invariants for Differential Equations

Differential Invariant



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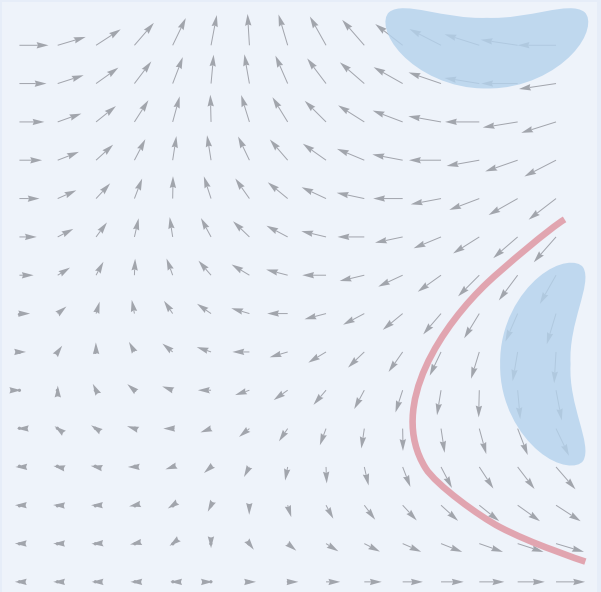

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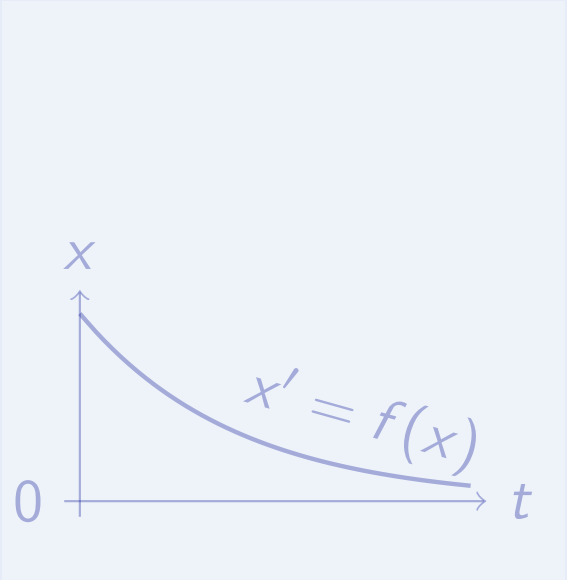

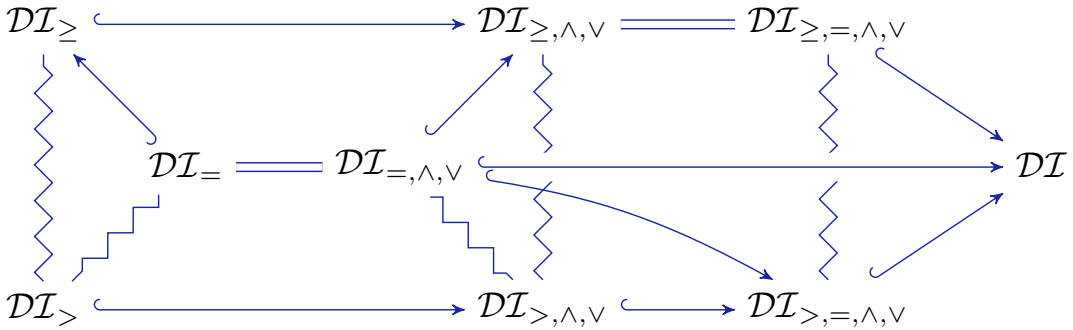

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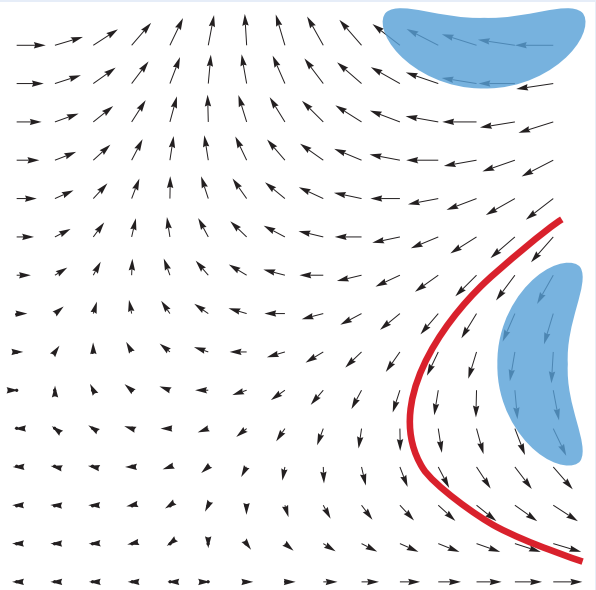

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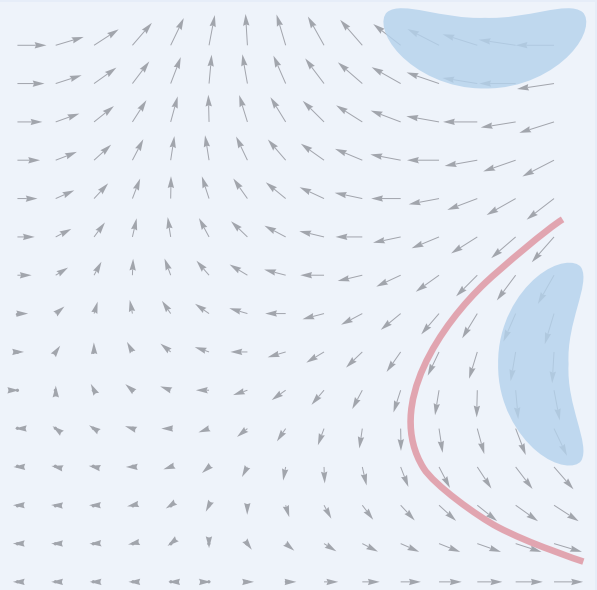

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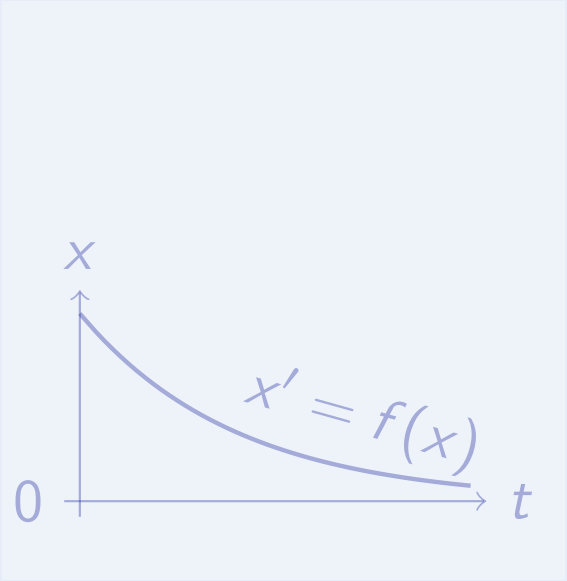

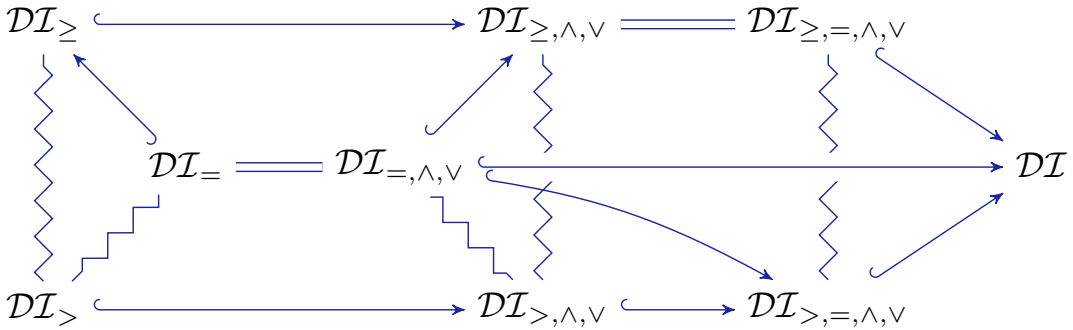

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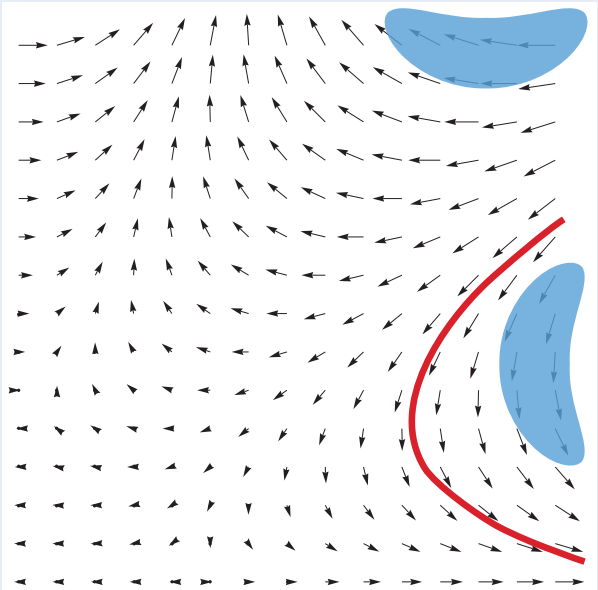

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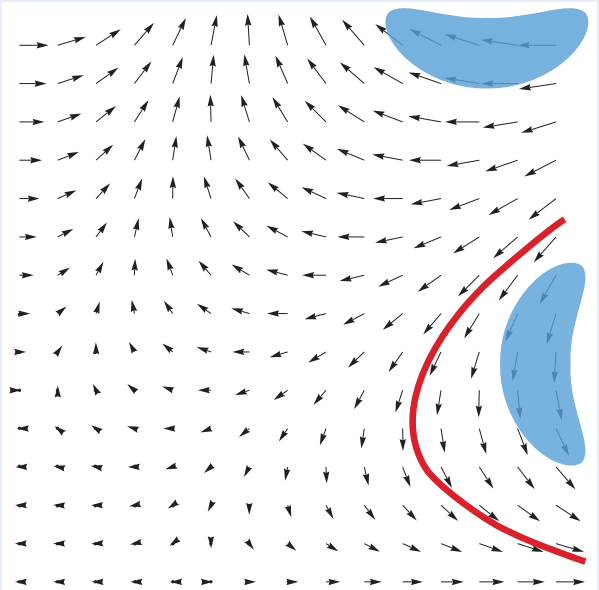

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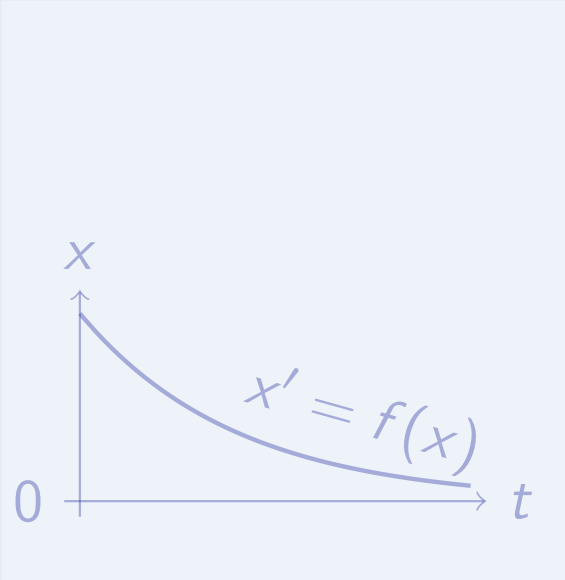

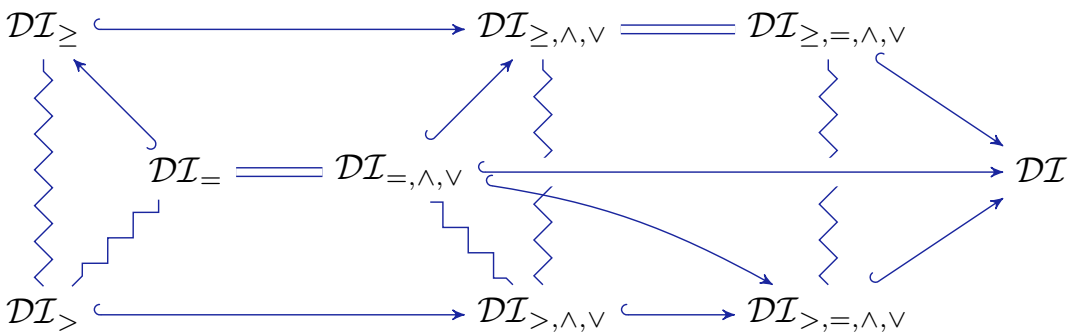

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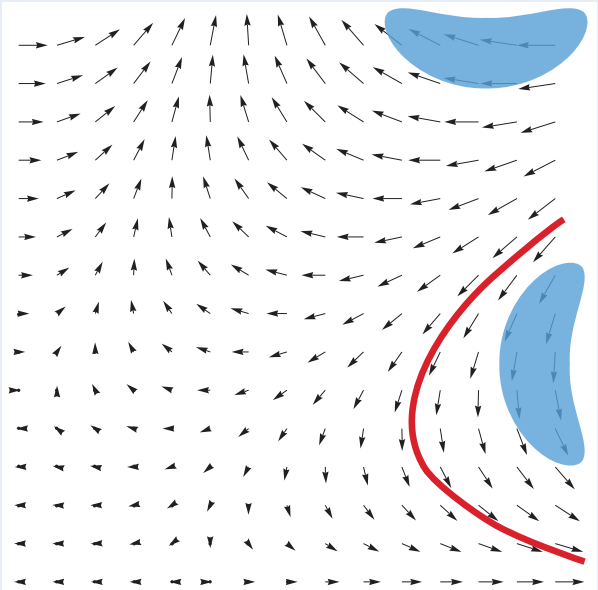

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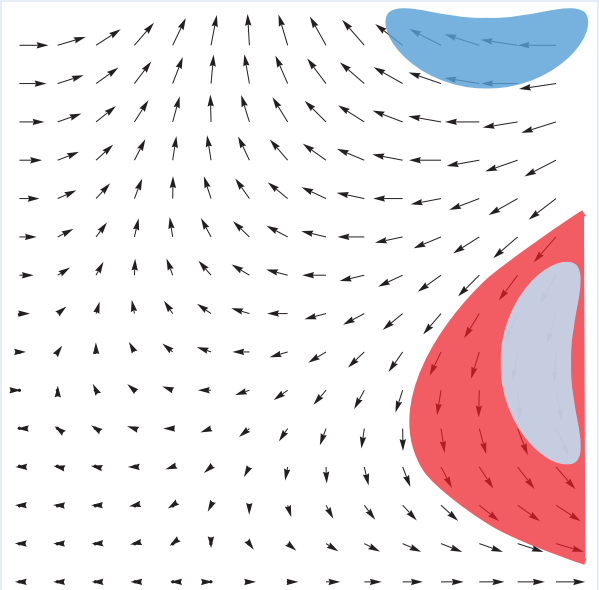

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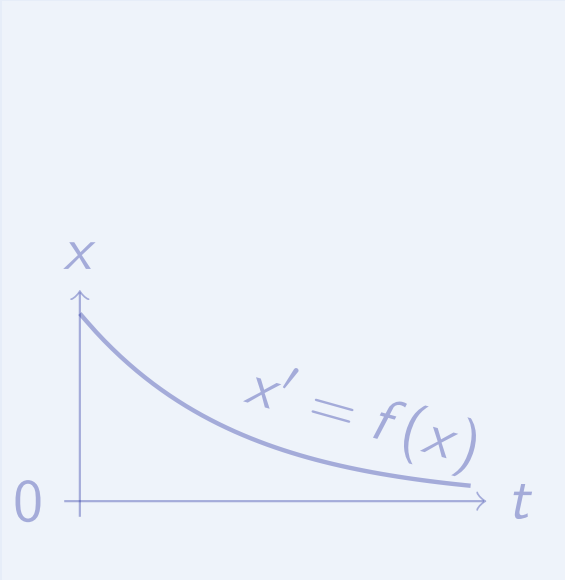

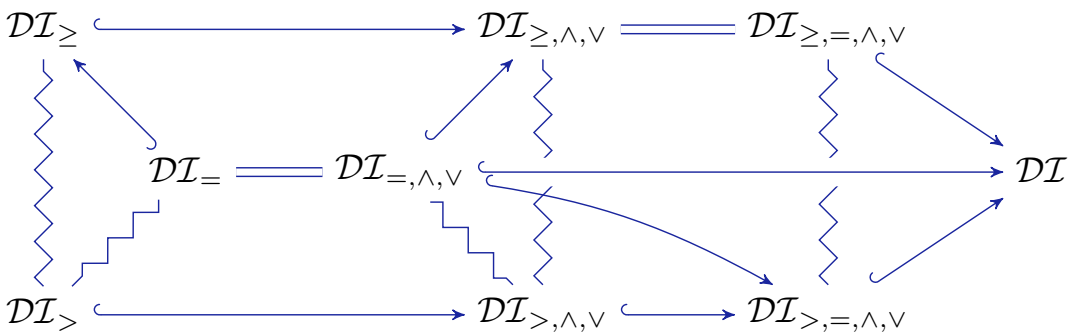

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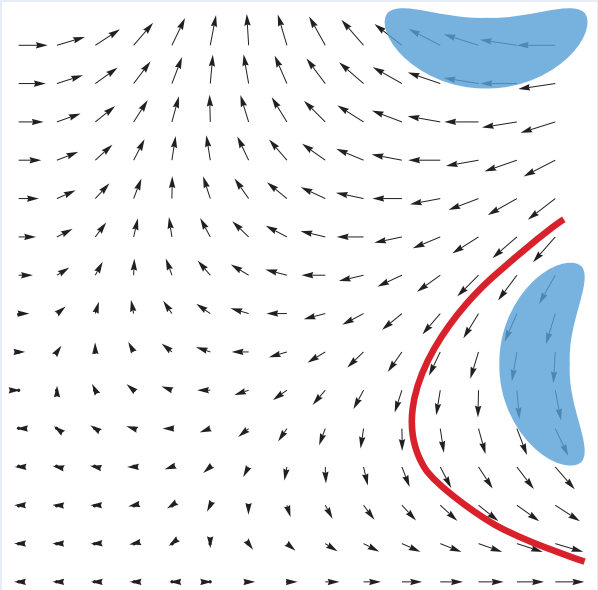

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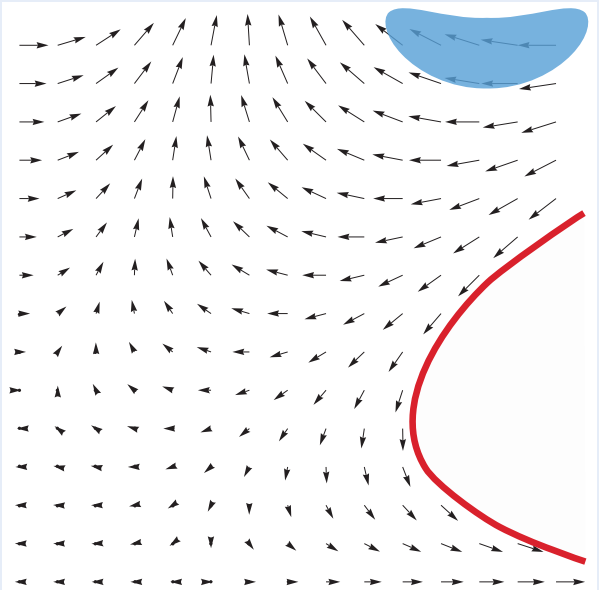

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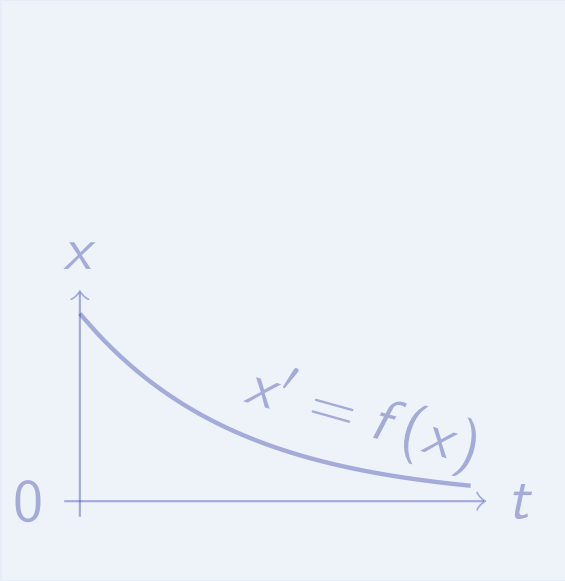

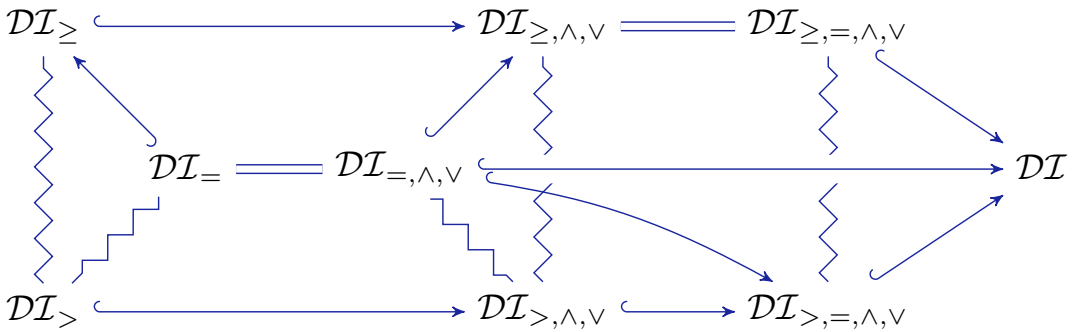

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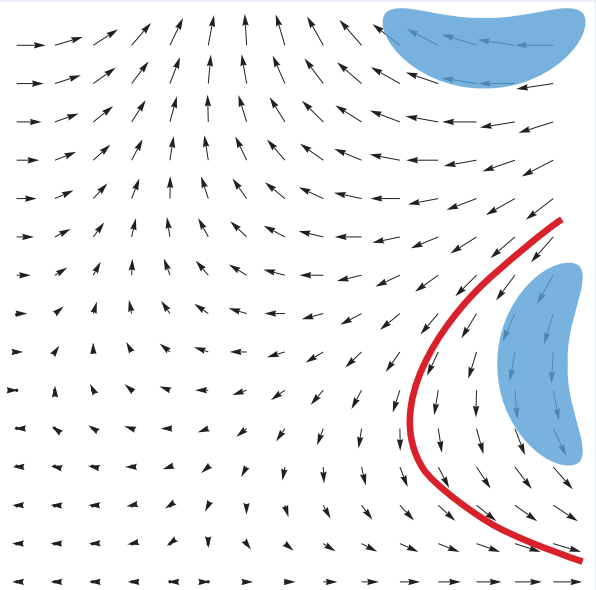

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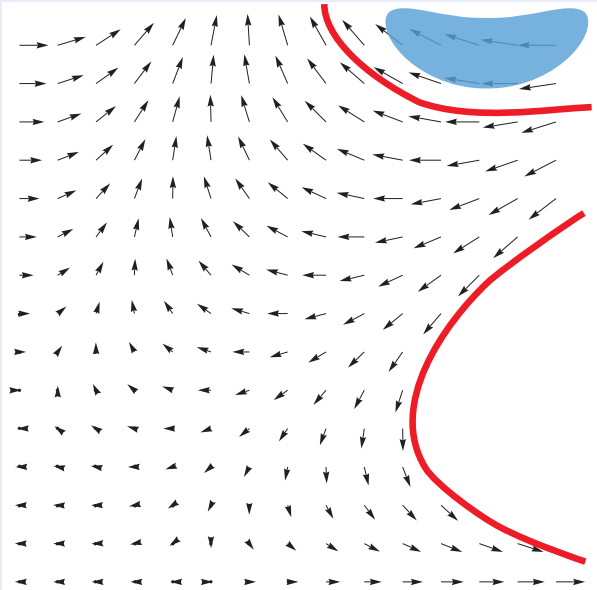

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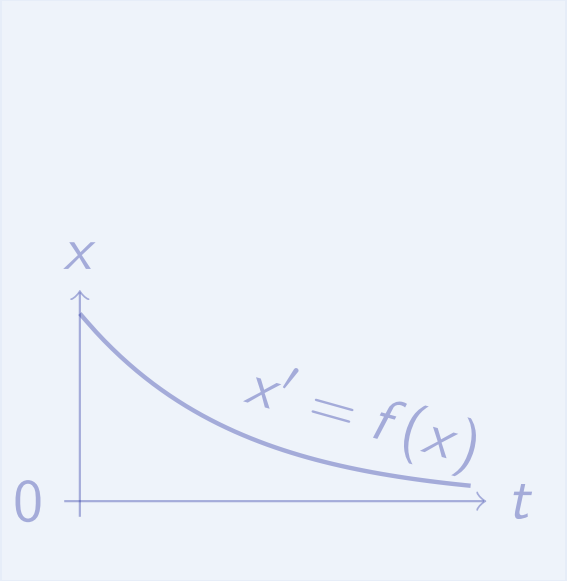

$$
\begin{tikzcd}
\mathcal{DI}_{\geq} \arrow[rr, hook] & & \mathcal{DI}_{\geq,\wedge,\vee} \arrow[r, equal] & \mathcal{DI}_{\geq,=,\wedge,\vee} \arrow[dr, squiggly] & \\
& \mathcal{DI}_{=} \arrow[ul, squiggly] \arrow[r, equal] \arrow[dl, squiggly] & \mathcal{DI}_{=,\wedge,\vee} \arrow[ul, hook] \arrow[dl, squiggly] \arrow[rr, hook] \arrow[drr, hook] & & \mathcal{DI} \\
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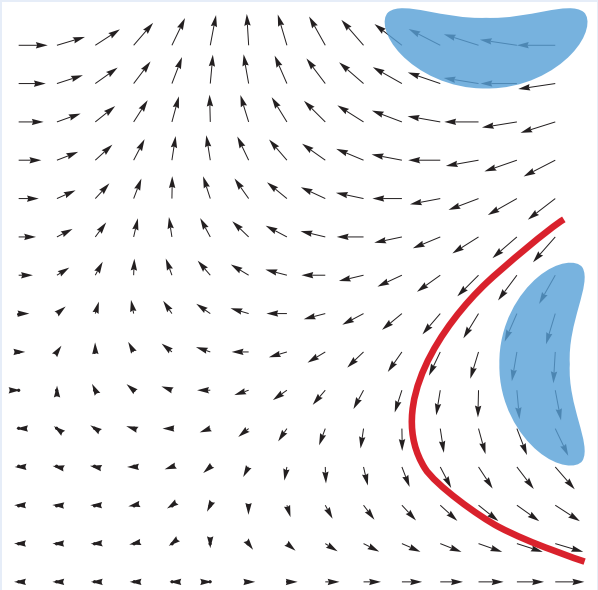
Differential Invariants for Differential Equations

Differential Invariant



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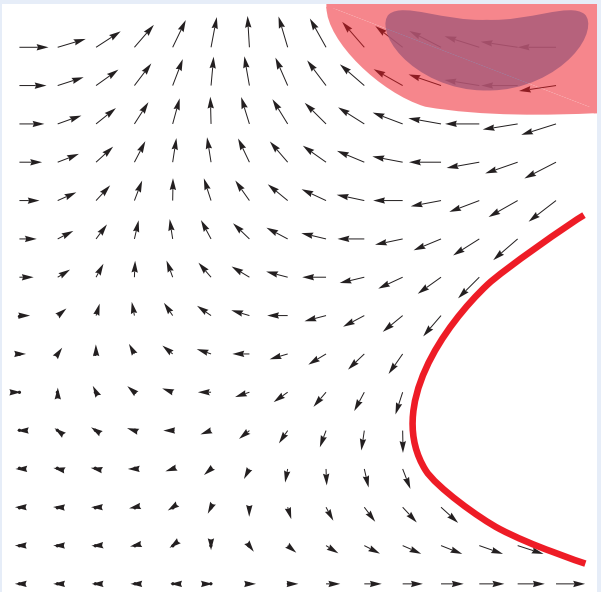

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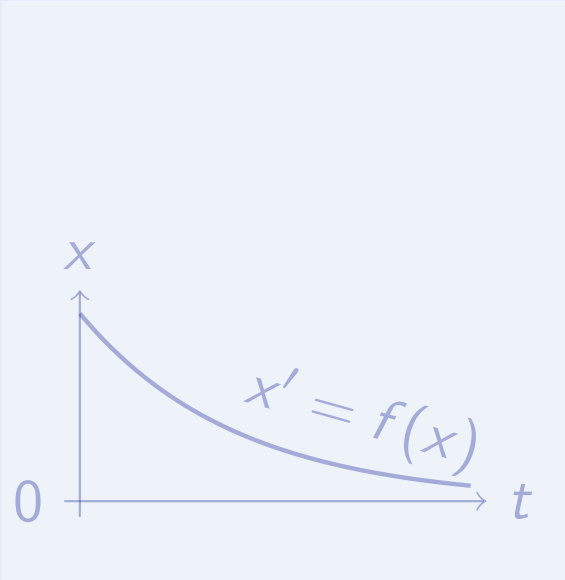

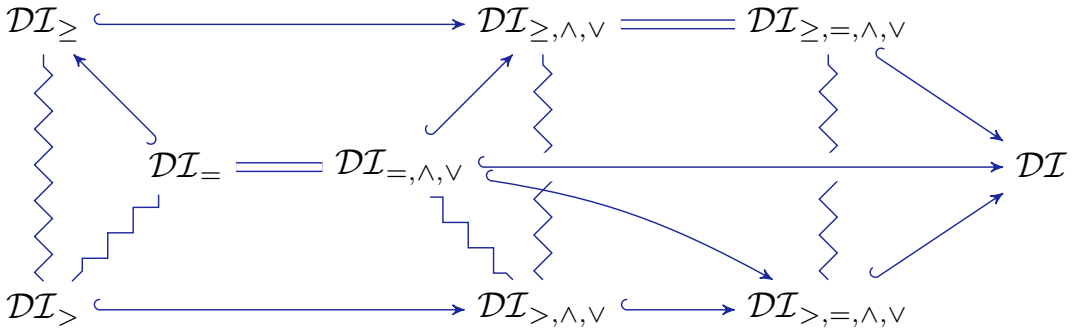

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Differential Invariants for Differential Equations

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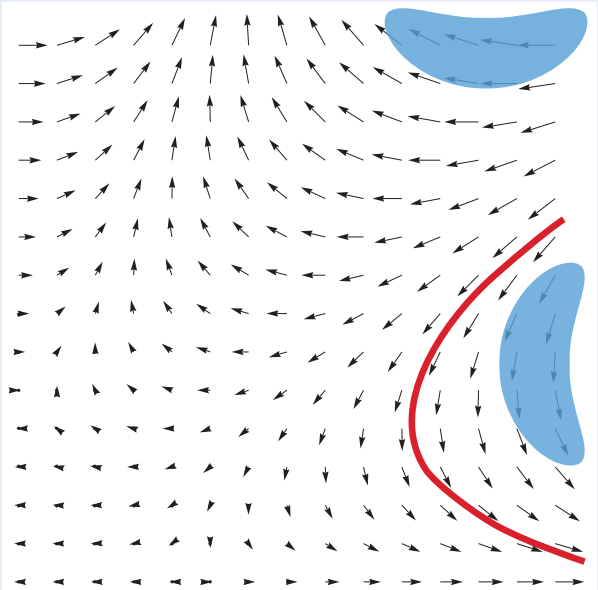

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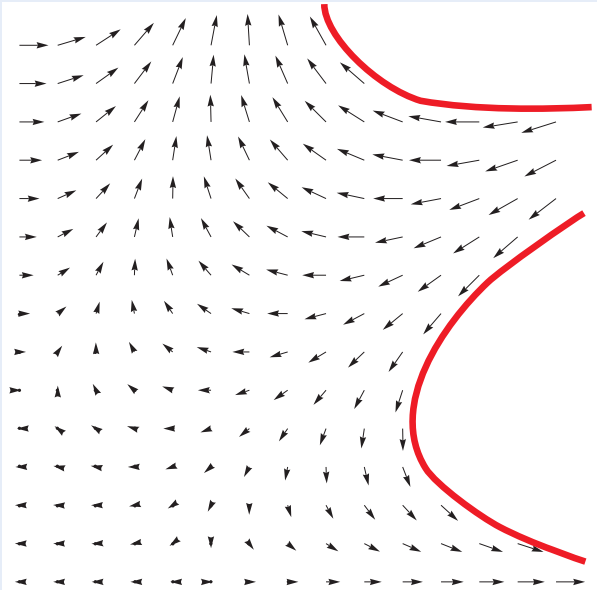

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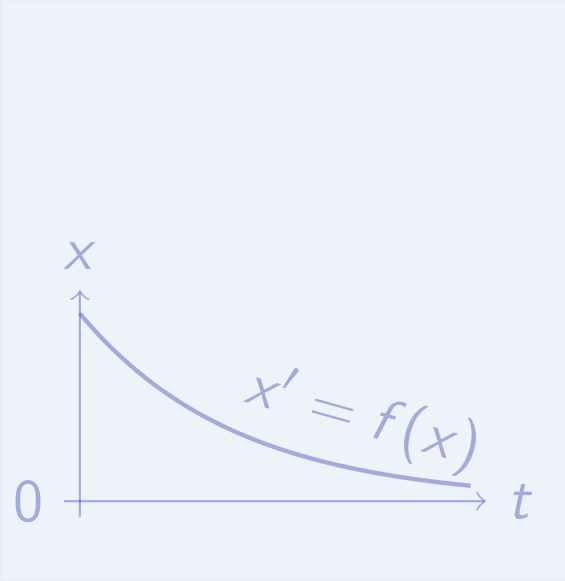

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& & & & \\
\mathcal{DI}_{>} \arrow[uuu, squiggly, bend left] \arrow[uu, squiggly] \arrow[uur, hook] \arrow[rr, hook] & & \mathcal{DI}_{>,\wedge,\vee} \arrow[r, hook] & \mathcal{DI}_{>,=,\wedge,\vee} \arrow[uur, hook] \arrow[uuu, squiggly, bend right] &
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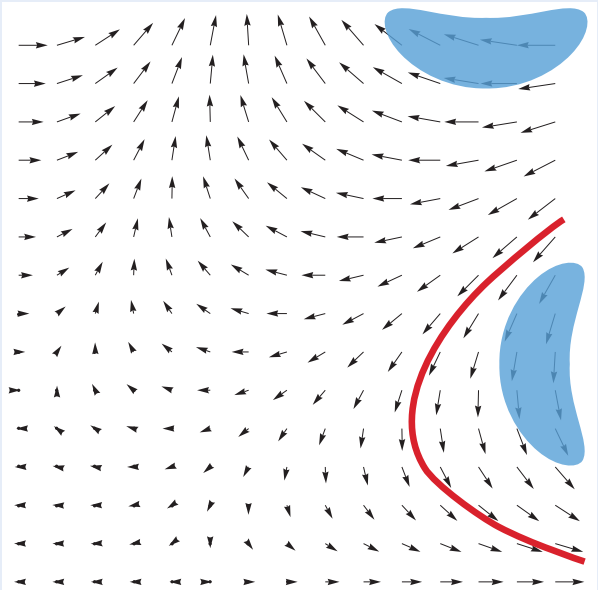

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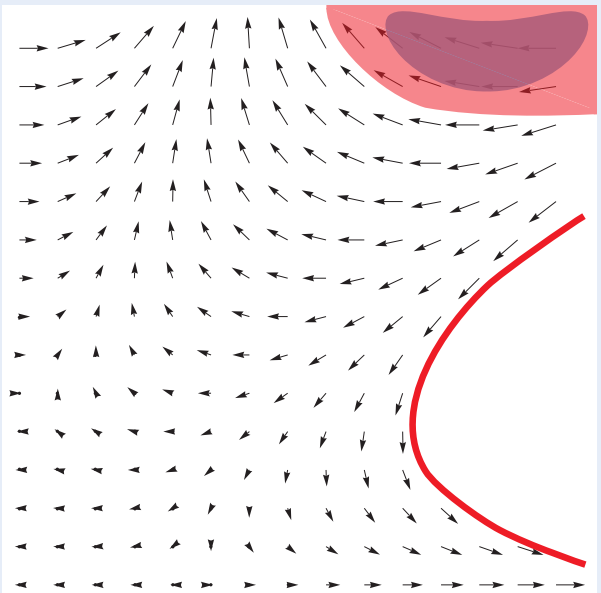

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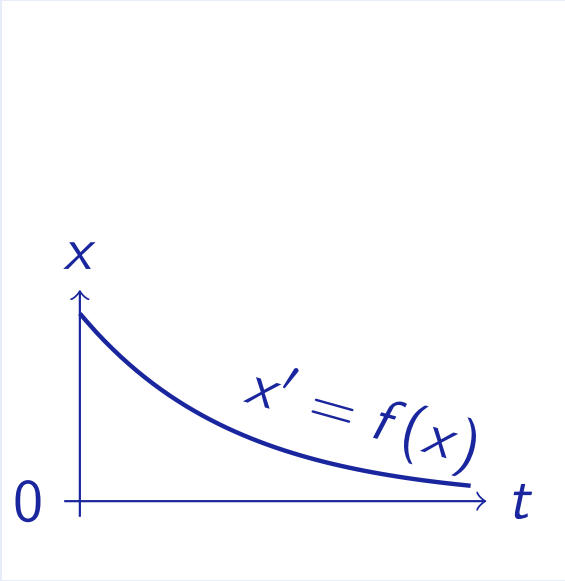

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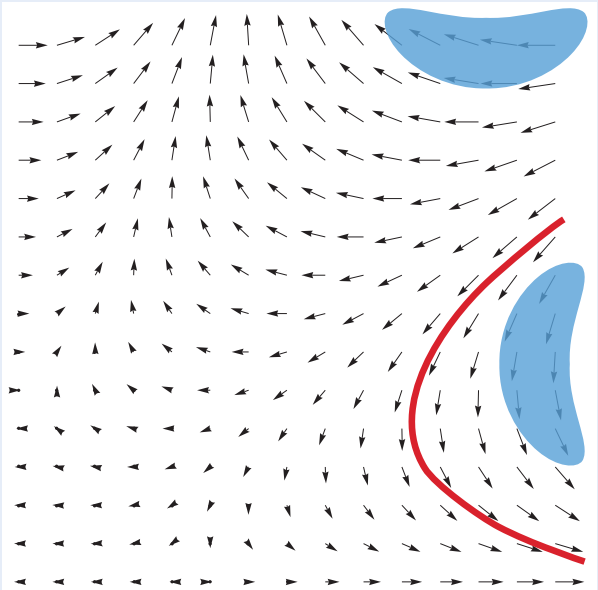

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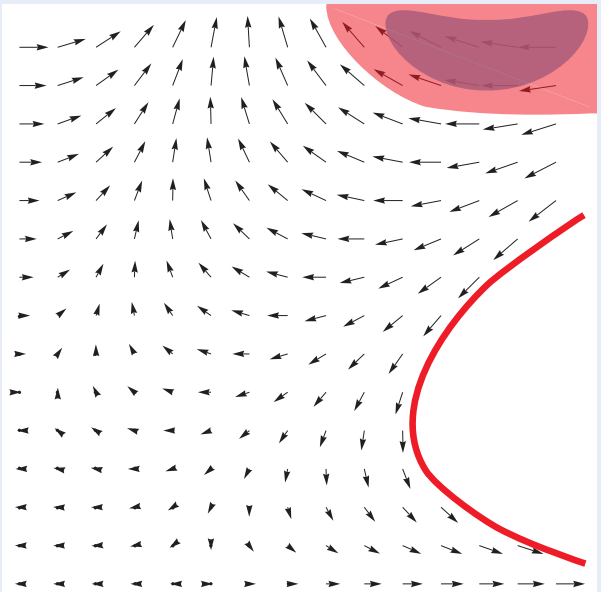

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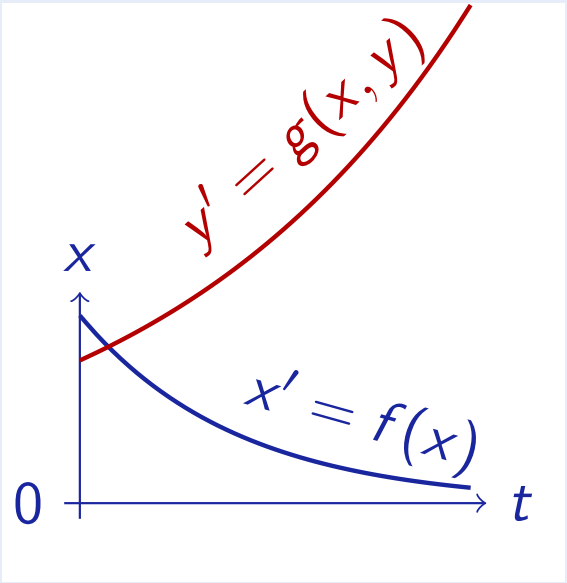

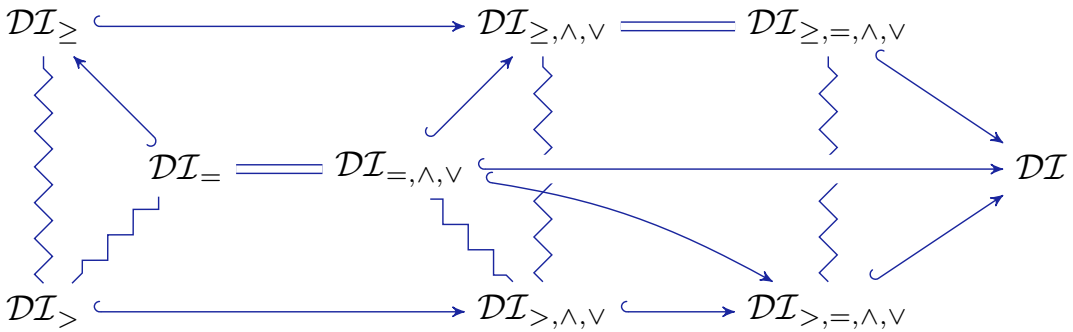

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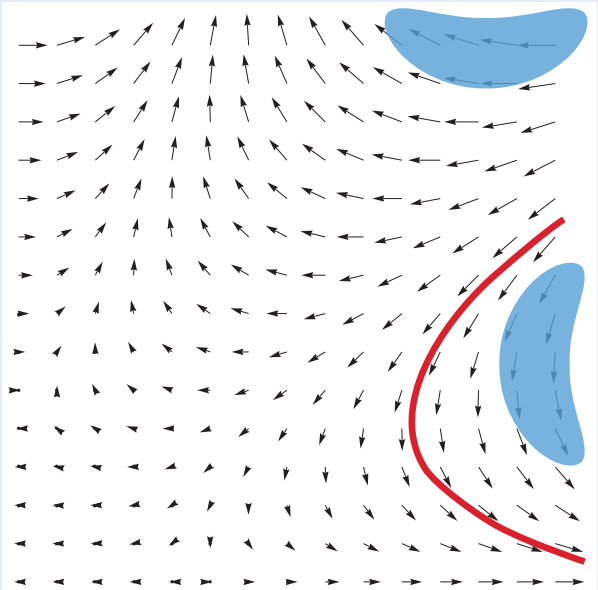

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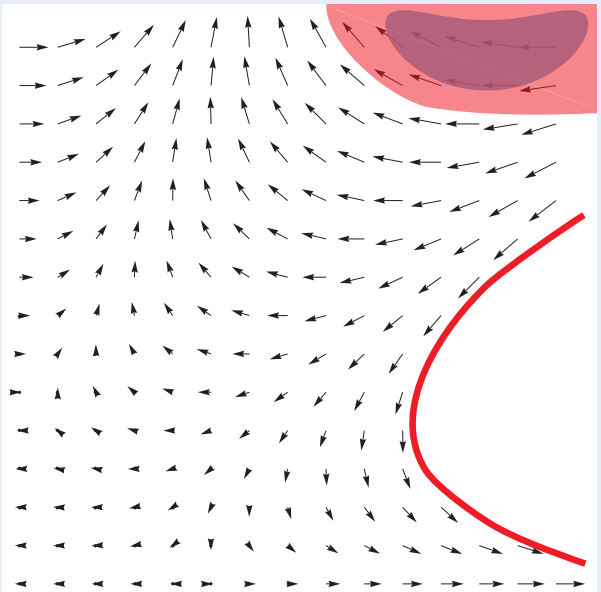
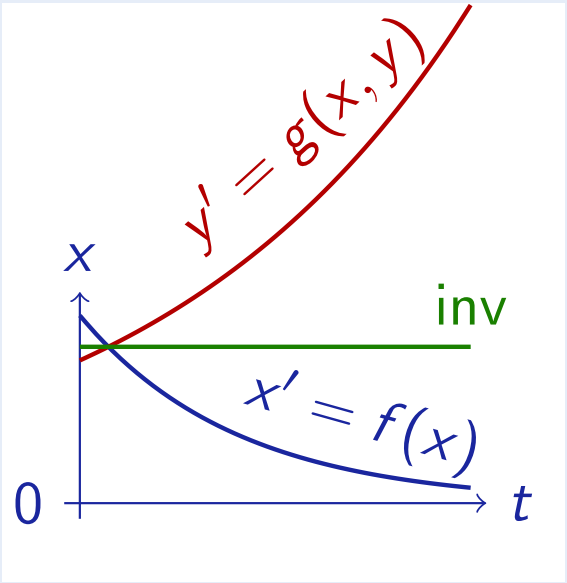
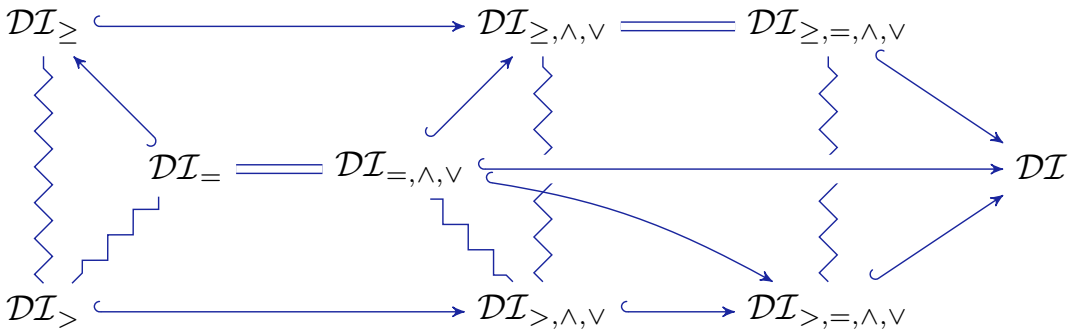


Differential Ghost





Logic

Provability theory

Math

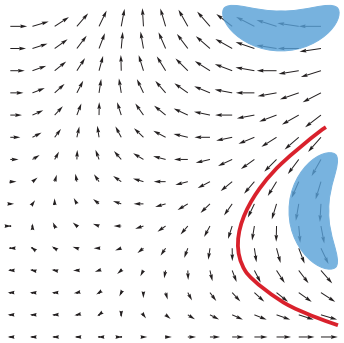
Characteristic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,LICS'12,ITP'12,JAR'16

Differential Invariants for Differential Equations

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$



JLogComput'10,LMCS'12, LICS'12,JAR'16

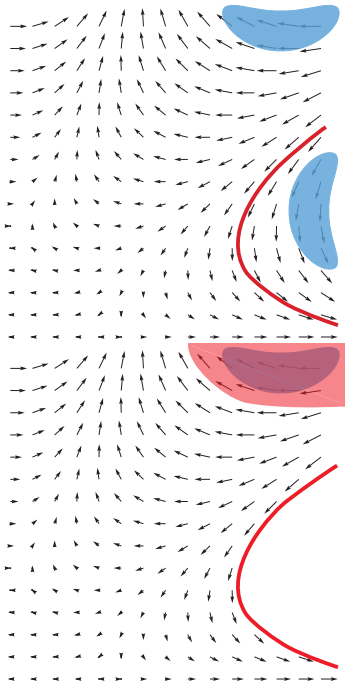
Differential Invariants for Differential Equations

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

Differential Cut

$$\frac{F \vdash [x' = f(x) \,\&\, Q]C \qquad F \vdash [x' = f(x) \,\&\, Q \wedge C]F}{F \vdash [x' = f(x) \,\&\, Q]F}$$



JLogComput'10,LMCS'12, LICS'12,JAR'16

Differential Invariants for Differential Equations

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$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

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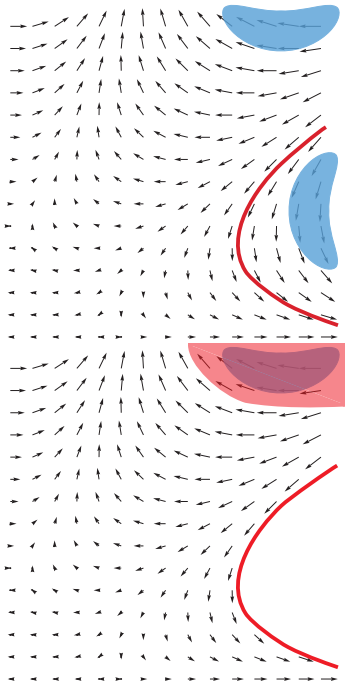
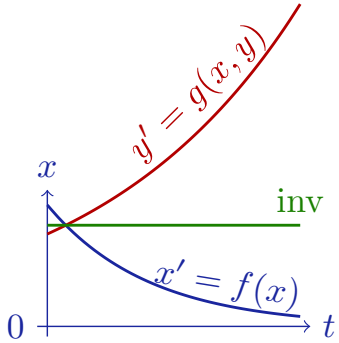
Differential Ghost

$$\frac{F \leftrightarrow \exists y\, G \qquad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$





JLogComput'10,LMCS'12, LICS'12,JAR'16

Differential Invariants for Differential Equations

Differential Invariant

$$\frac{Q \vdash [x' := f(x)](F)'}{F \vdash [x' = f(x) \,\&\, Q]F}$$

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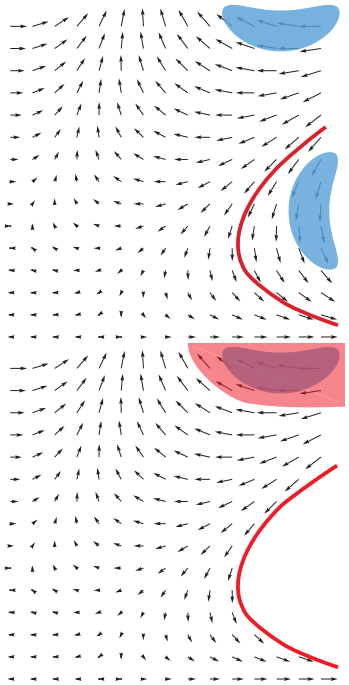
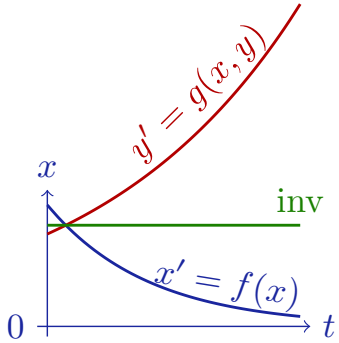
Differential Ghost

$$\frac{F \leftrightarrow \exists y\, G \qquad G \vdash [x' = f(x), y' = g(x,y) \,\&\, Q]G}{F \vdash [x' = f(x) \,\&\, Q]F}$$

if new $y' = g(x,y)$ has a global solution





JLogComput'10,LMCS'12, LICS'12,JAR'16

Differential Equation Axioms & Differential Axioms

DW $[x' = f(x) \,\&\, Q]Q$

DC $\big([x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q \wedge R]P\big)$
$\leftarrow [x' = f(x) \,\&\, Q]R$

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

DI $\big([x' = f(x) \,\&\, Q]P \leftrightarrow [?Q]P\big) \leftarrow [x' = f(x) \,\&\, Q](P)'$

DG $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$

DS $[x' = c() \,\&\, Q]P \leftrightarrow \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\; q(x{+}c()s)) \rightarrow [x := x{+}c()t]P\big)$

$[':=]$ $[x' := e]p(x') \leftrightarrow p(e)$

$+'$ $(e + k)' = (e)' + (k)'$

$\cdot'$ $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$

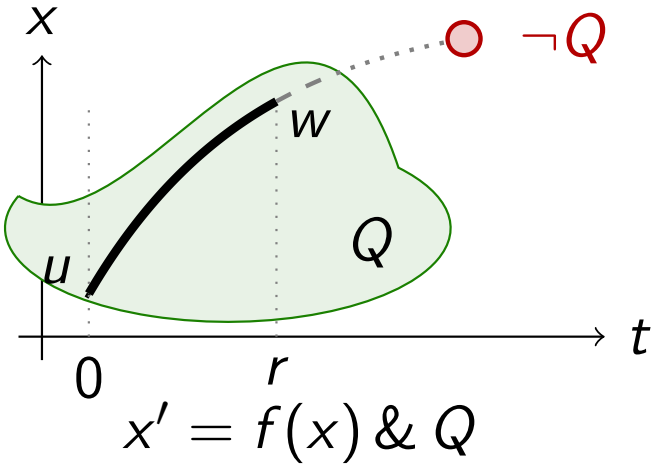
$\circ'$ $[y := g(x)][y' := 1]\big((f(g(x)))' = (f(y))' \cdot (g(x))'\big)$

CADE'15

Differential Equation Axioms

Axiom (Differential Weakening) (CADE'15)

DW $[x' = f(x) \,\&\, Q]Q$



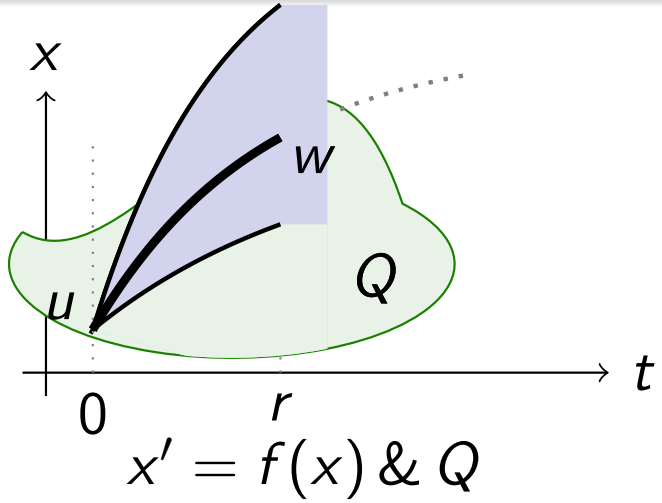
Differential equations cannot leave their evolution domains. Implies:

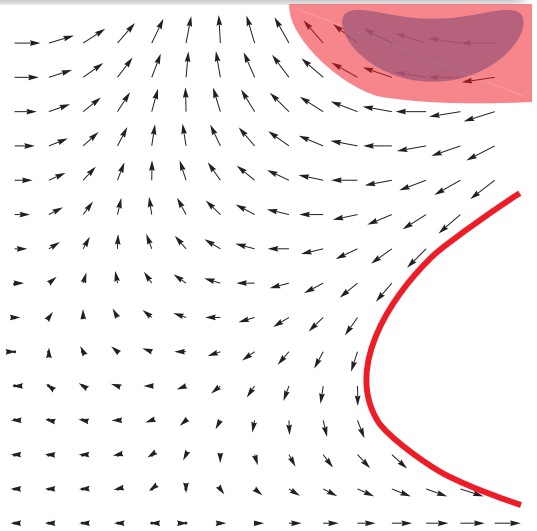
$$[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q](Q \rightarrow P)$$

Differential Equation Axioms

Axiom (Differential Cut) (CADE'15)

DC $\quad ([x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q \wedge R]P)$
$\quad \leftarrow [x' = f(x) \,\&\, Q]R$



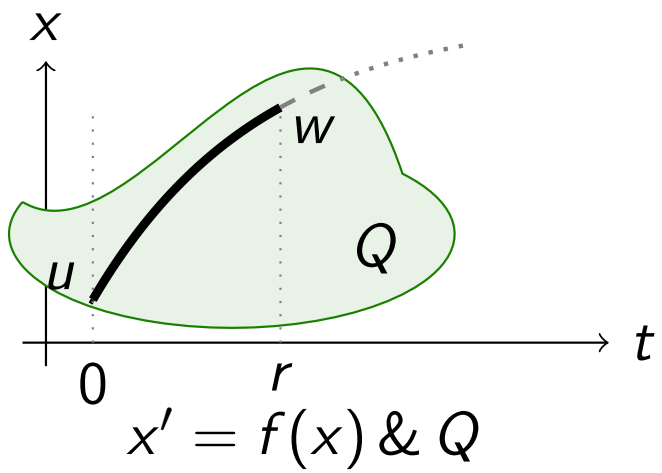


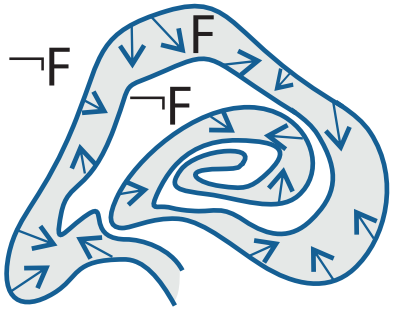
DC is a cut for differential equations.
DC is a differential modal modus ponens K.
Can't leave R, then might as well restrict state space to R.

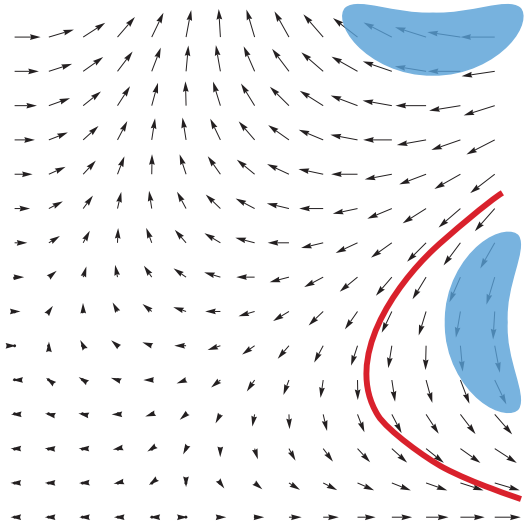
Differential Equation Axioms

Axiom (Differential Invariant) (CADE'15)

DI $\big([x' = f(x) \,\&\, Q]P \leftrightarrow [?Q]P\big) \leftarrow [x' = f(x) \,\&\, Q](P)'$







Differential invariant: if P true now and if differential $(P)'$ true always

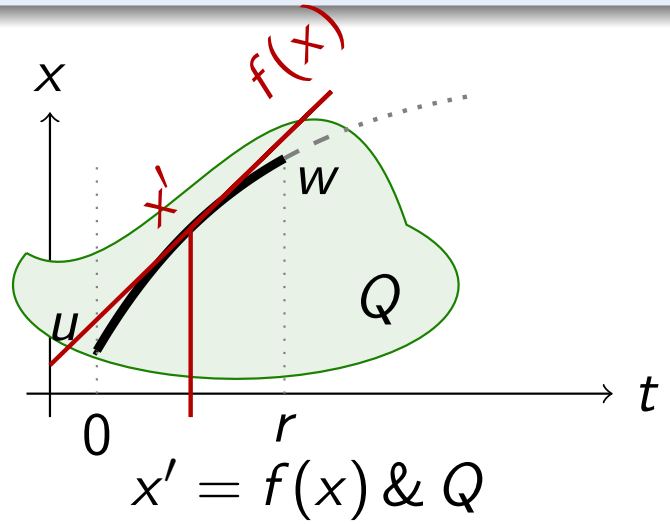
What's the differential of a formula???

What's the meaning of a differential term ... in a state???

Differential Equation Axioms

Axiom (Differential Effect) (CADE'15)

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$



Effect of differential equation on differential symbol x'

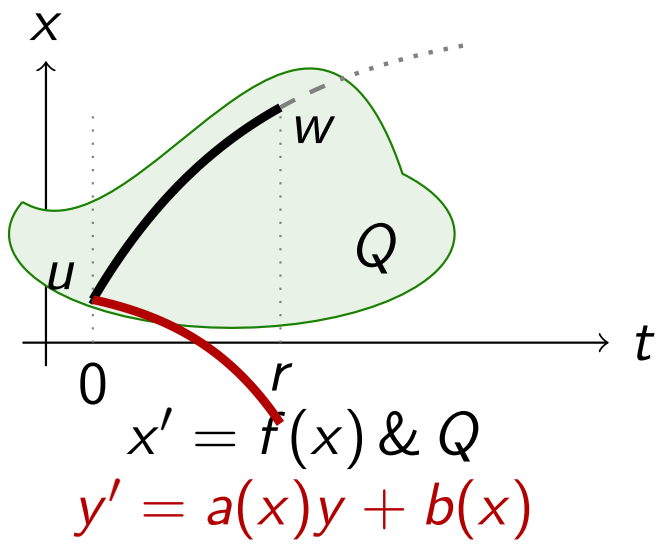
$[x' := f(x)]$ instantly mimics continuous effect $[x' = f(x)]$ on x'

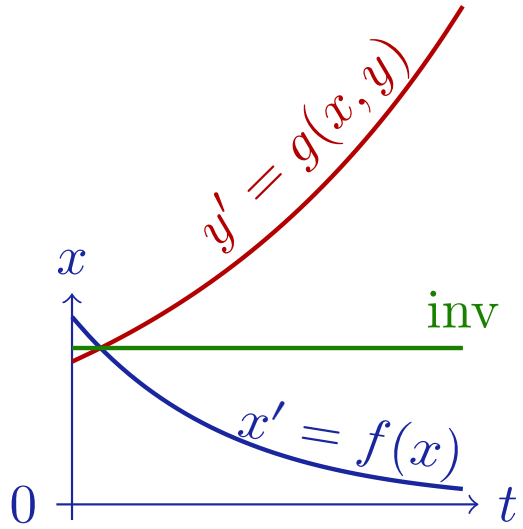
$[x' := f(x)]$ selects vector field $x' = f(x)$ for subsequent differentials

Differential Equation Axioms

Axiom (Differential Ghost) (CADE'15)

DG $[x' = f(x) \,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, Q]P$





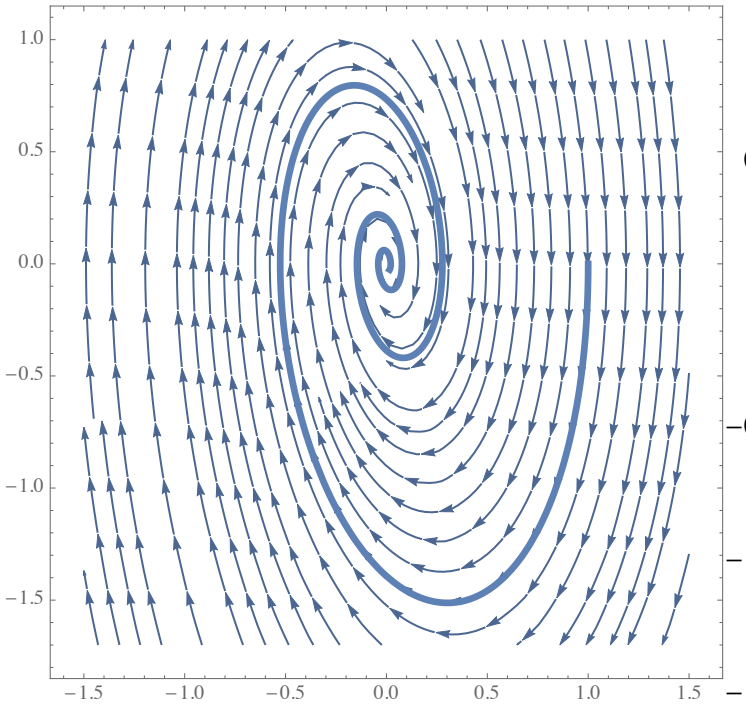
Differential ghost/auxiliaries: extra differential equations that exist

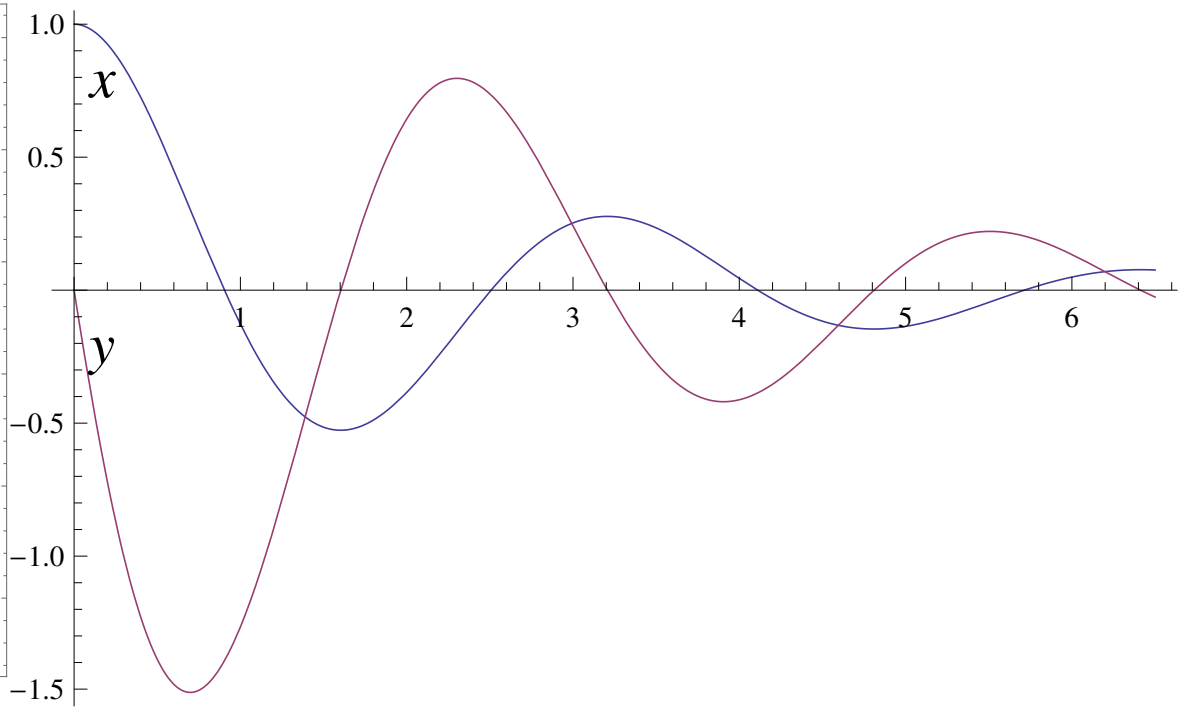
Can cause new invariants

"Dark matter" counterweight to balance conserved quantities

Differential Invariants for Differential Equations

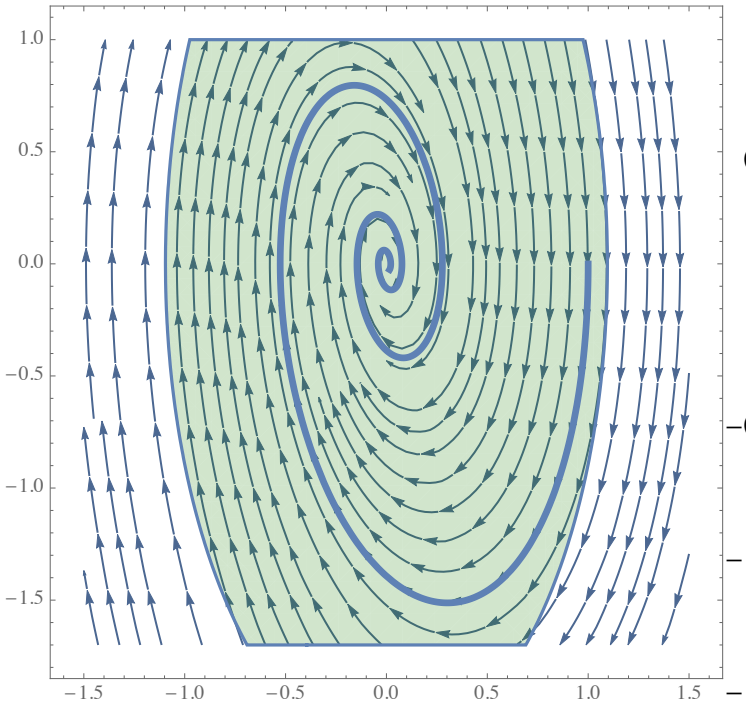
$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2$$

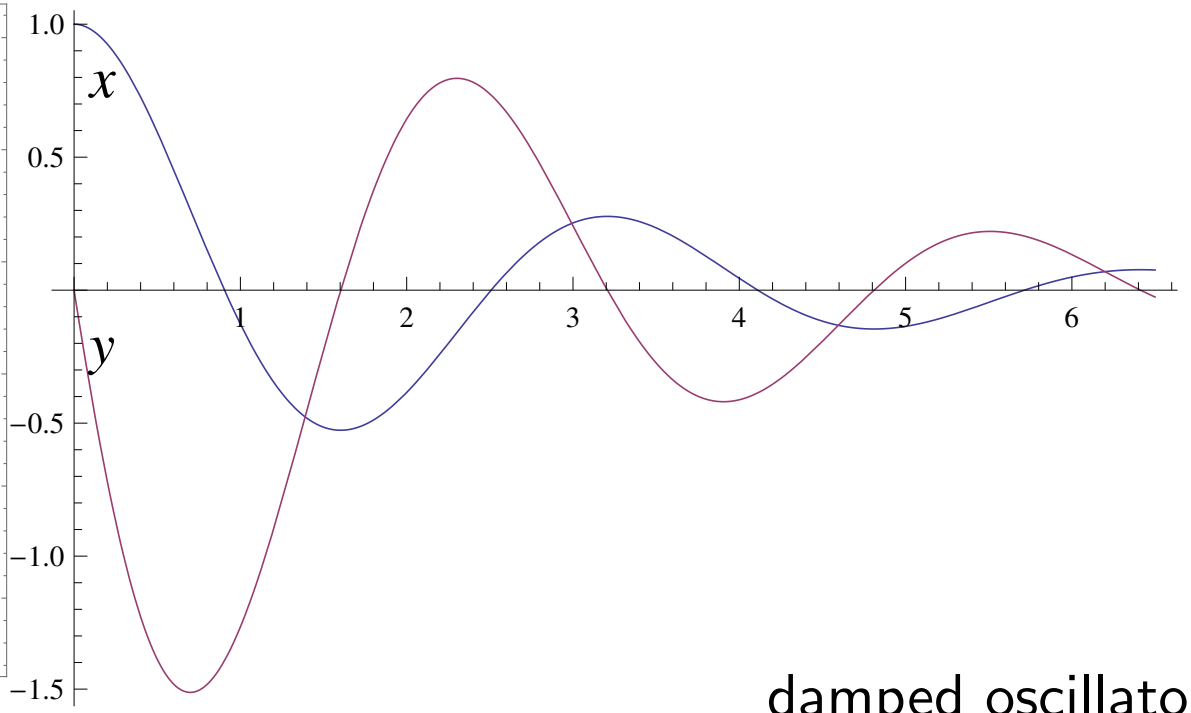




Differential Invariants for Differential Equations

$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2$$



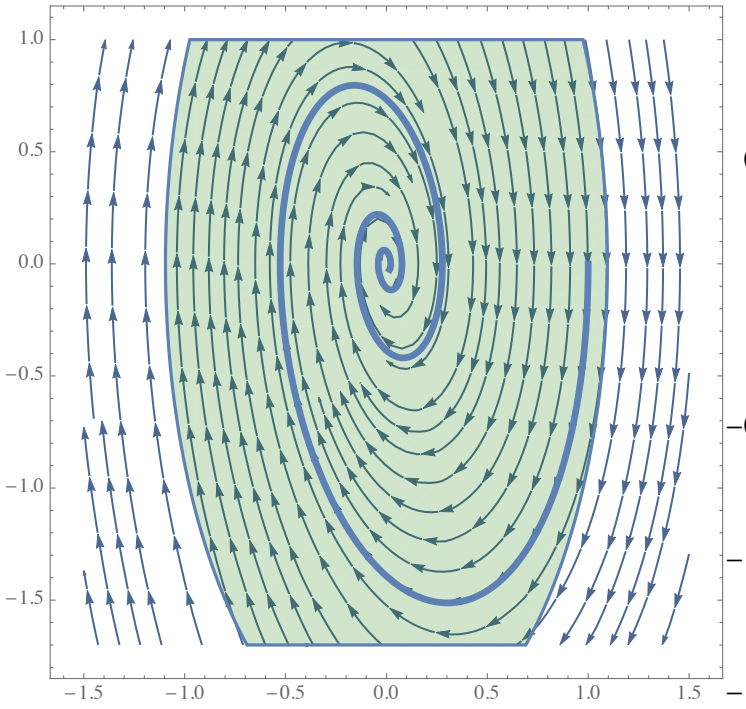


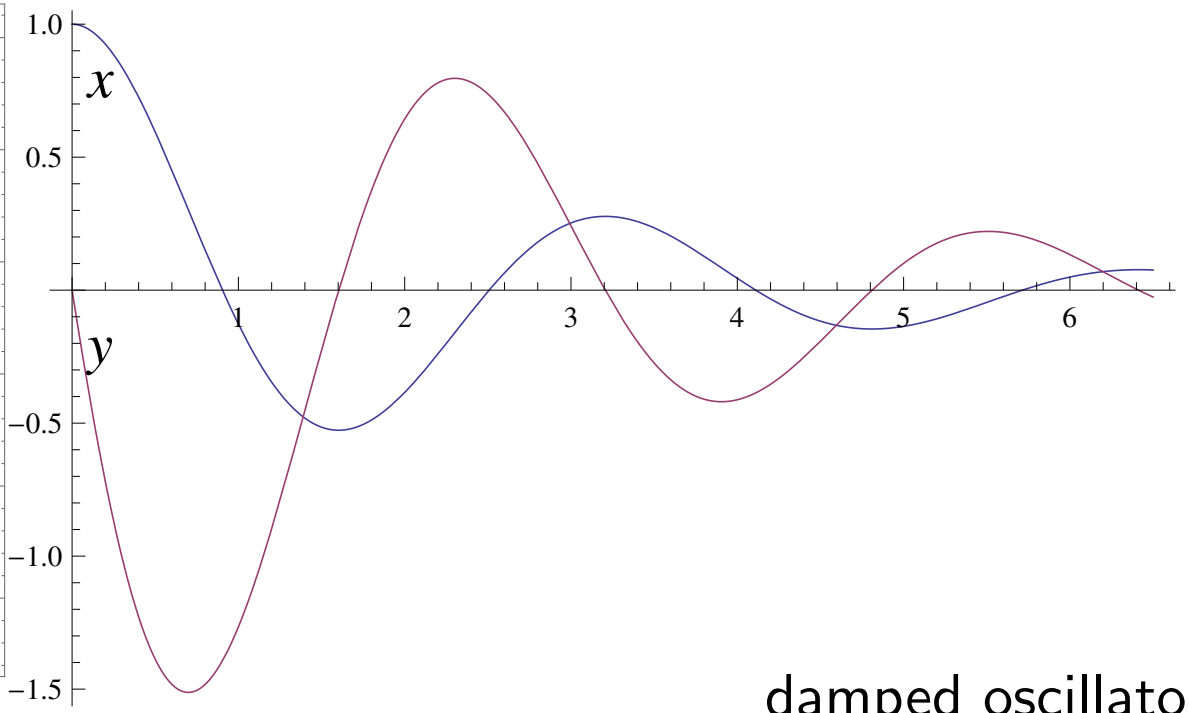
damped oscillator

Differential Invariants for Differential Equations

$$\omega \geq 0 \wedge d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega \geq 0 \wedge d \geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$





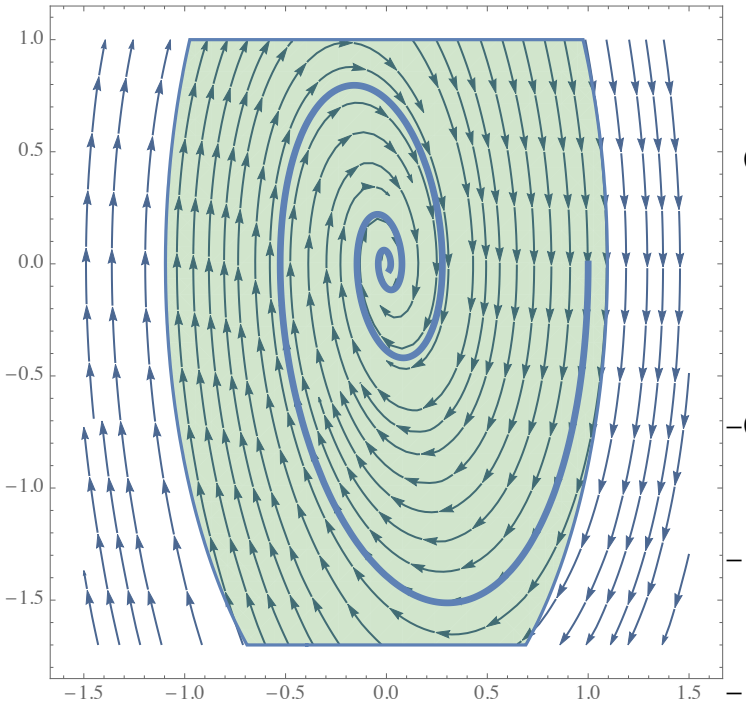
damped oscillator

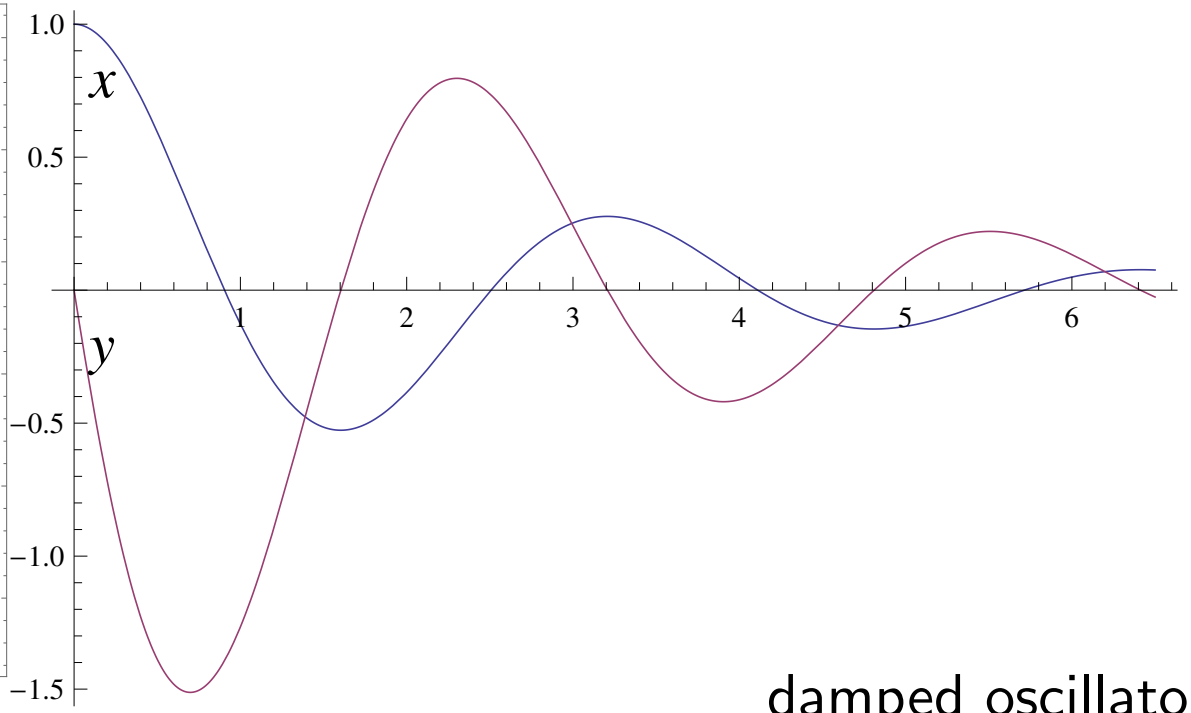
Differential Invariants for Differential Equations

$$\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}-\omega^2 x - 2d\omega y]\, 2\omega^2 xx' + 2yy' \leq 0$$

$$\omega^2x^2{+}y^2{\leq}c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2x^2{+}y^2{\leq}c^2$$



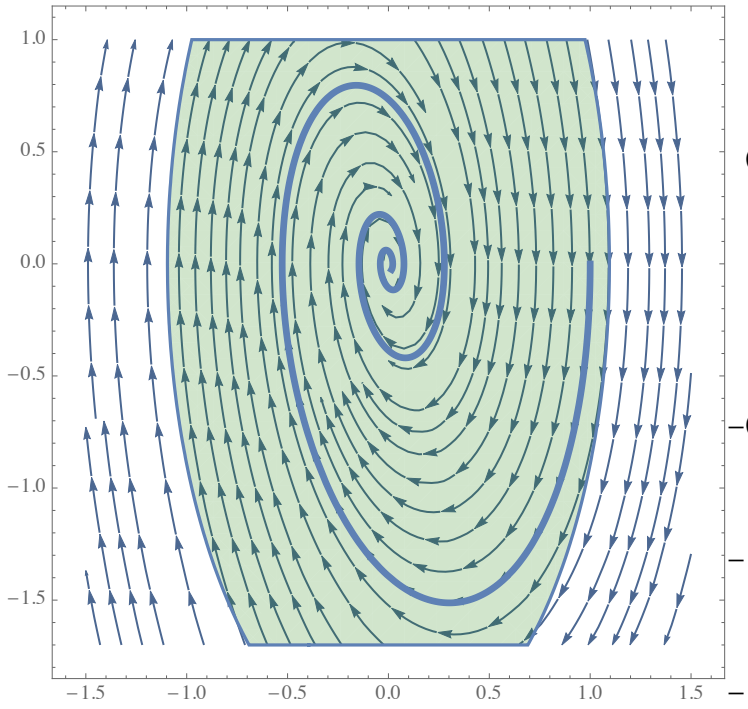


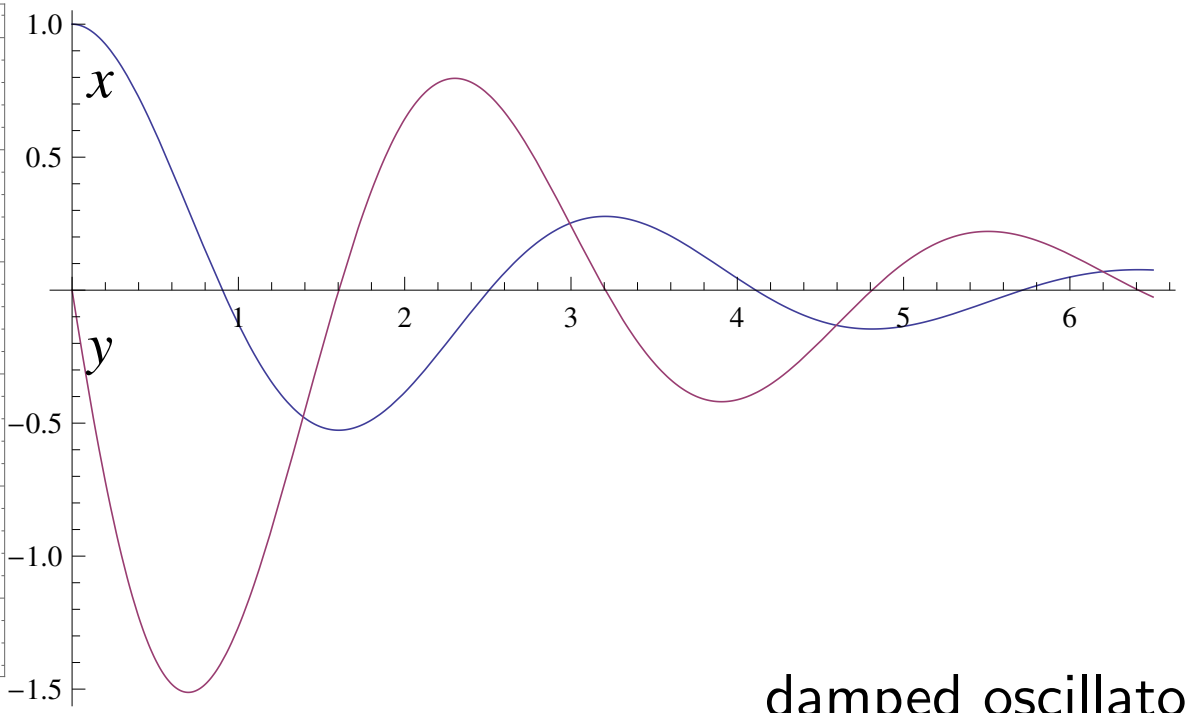
damped oscillator

Differential Invariants for Differential Equations

$$\frac{*}{\omega{\geq}0 \land d{\geq}0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0}$$

$$\frac{\omega{\geq}0 \land d{\geq}0 \vdash [x'{:=}y][y'{:=}{-}\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \leq 0}{\omega^2x^2{+}y^2{\leq}c^2 \vdash [x' = y, y' = -\omega^2x - 2d\omega y \,\&\, \omega{\geq}0 \land d{\geq}0]\, \omega^2x^2{+}y^2{\leq}c^2}$$





damped oscillator

Differential Invariants for Differential Equations

$$*$$

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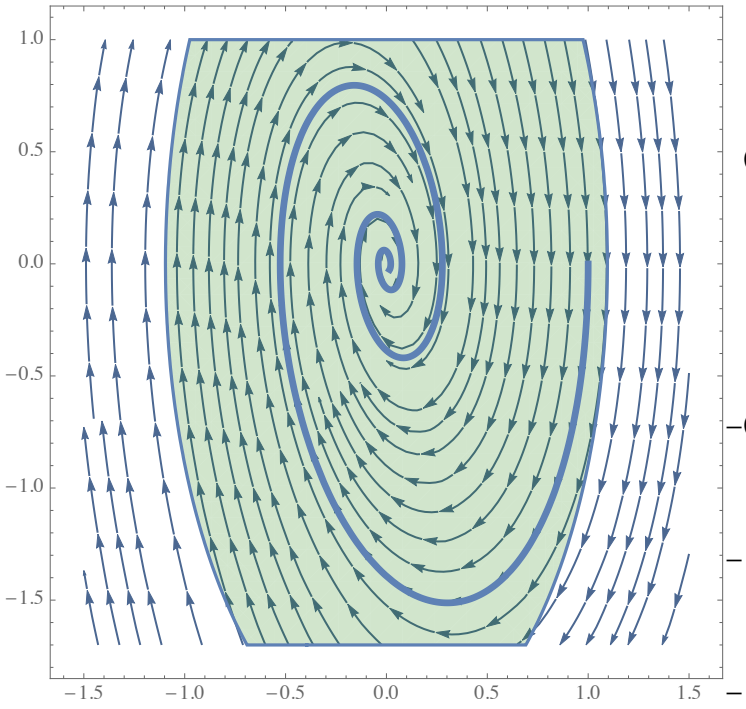
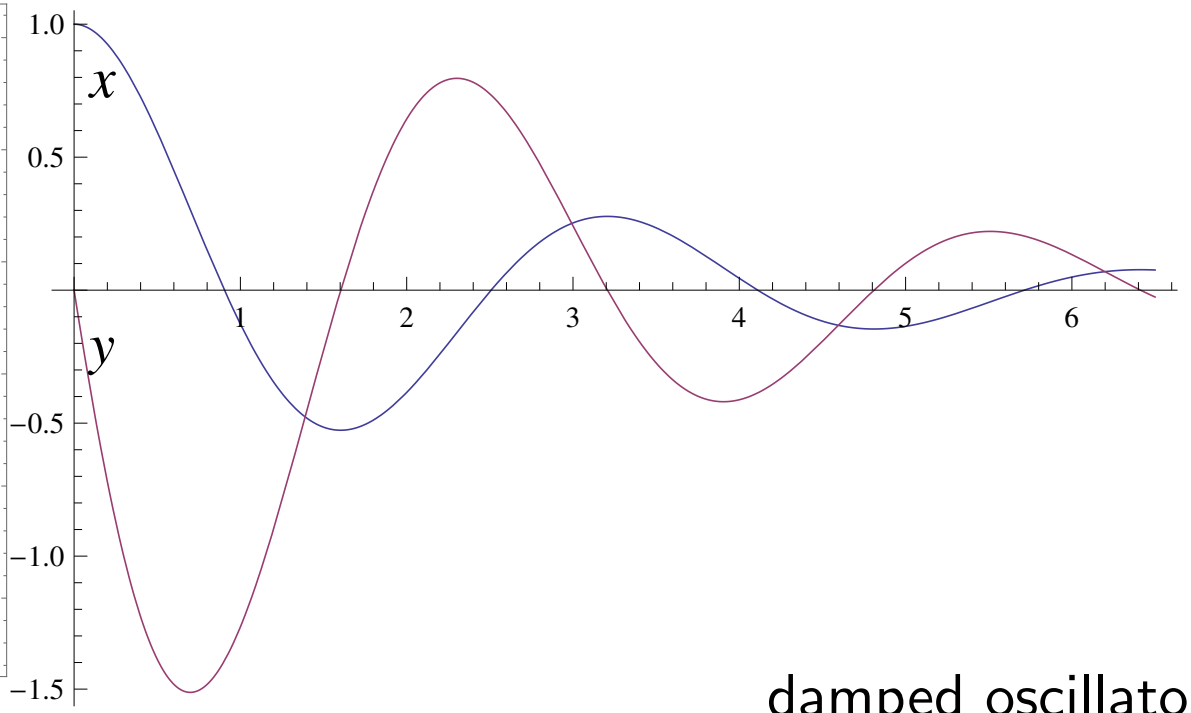




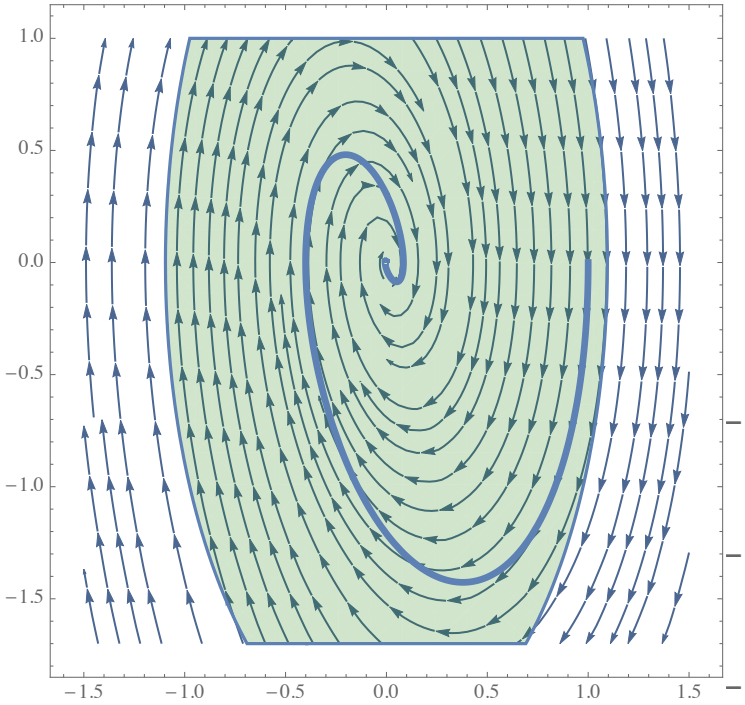
damped oscillator

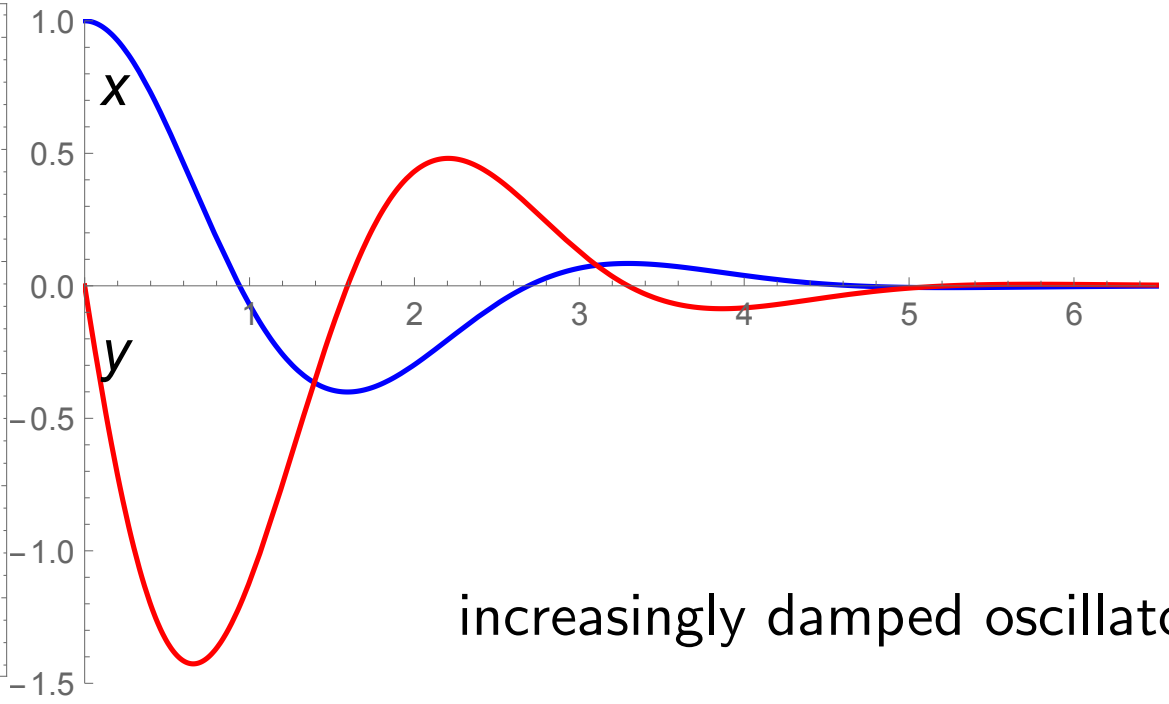
Differential Cuts for Differential Equations

$$\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2$$

Differential Cuts for Differential Equations

$$\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2$$





increasingly damped oscillator

Differential Cuts for Differential Equations

$$\frac{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

increasingly damped oscillator

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$$\frac{}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$\frac{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

$$\frac{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

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$$\frac{\dfrac{\dfrac{}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2$$

$$\omega^2x^2+y^2\leq c^2 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2$$

DC

$$\frac{\frac{\frac{*}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}}{d\geq 0 \vdash [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$\omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0$$

$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2$$

$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2$$

$$\omega\geq 0 \vdash 7\geq 0 \quad (*)$$

$$\omega\geq 0 \vdash [d':=7]\, d'\geq 0$$

$$d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$\omega\geq 0\wedge d\geq 0 \vdash 2\omega^2xy + 2y(-\omega^2x - 2d\omega y) \leq 0$$

$$\omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2x - 2d\omega y]\, 2\omega^2xx' + 2yy' \leq 0$$

$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0\wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2$$

$$\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2x-2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2$$

$$\frac{\frac{\frac{*}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\, d'\geq 0}}{d\geq 0 \vdash [x' = y, y' = -\omega^2x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$\frac{\frac{\frac{\frac{*}{\omega\geq 0\wedge d\geq 0 \vdash 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq 0\wedge d\geq 0 \vdash [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq 0\wedge d\geq 0]\,\omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

$$\frac{\frac{\frac{*}{\omega\geq 0 \vdash 7\geq 0}}{\omega\geq 0 \vdash [d':=7]\,d'\geq 0}}{d\geq 0 \vdash [x'=y,y'=-\omega^2x-2d\omega y,d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

$$
\begin{array}{c}
* \\
\hline
\omega\geq 0 \wedge d\geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \\
\hline
\omega\geq 0 \wedge d\geq 0 \vdash [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 xx' + 2yy' \leq 0 \\
\hline
\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \wedge d\geq 0]\, \omega^2x^2+y^2\leq c^2 \\
\hline
\omega^2x^2+y^2\leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2
\end{array}
$$

$$
\begin{array}{c}
* \\
\hline
\omega\geq 0 \vdash 7\geq 0 \\
\hline
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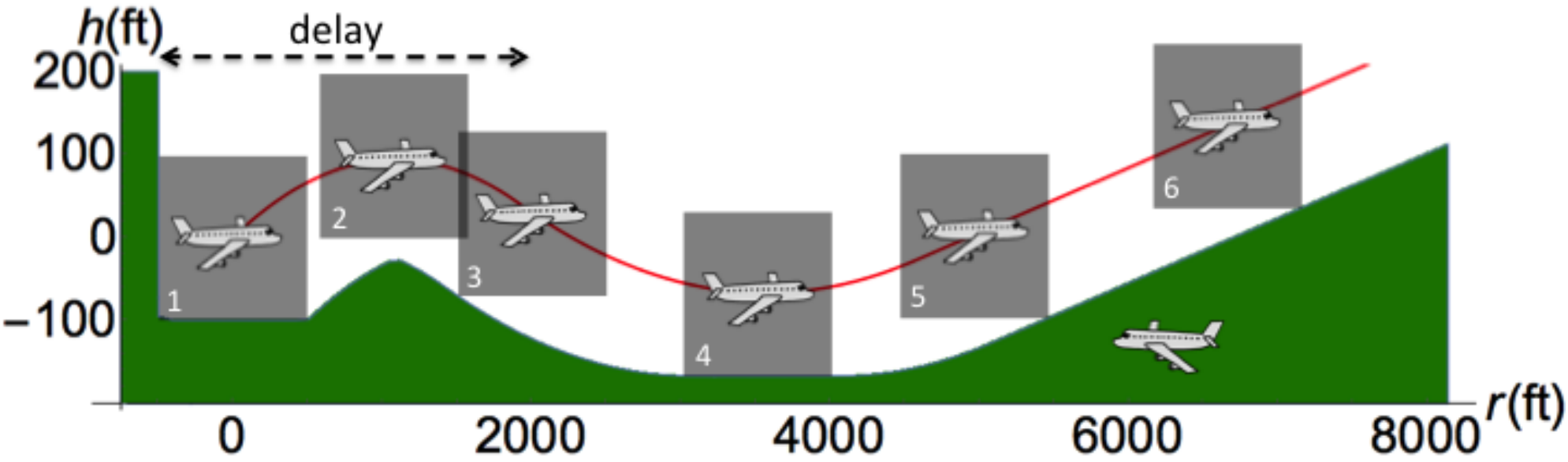
\hline
d\geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0
\end{array}
$$

Could repeatedly diffcut in formulas to help the proof

Airborne Collision Avoidance System ACAS X: Verify

- Developed by the FAA to replace current TCAS in aircraft
- Approximately optimizes Markov Decision Process on a grid
- Advisory from lookup tables with numerous 5D interpolation regions

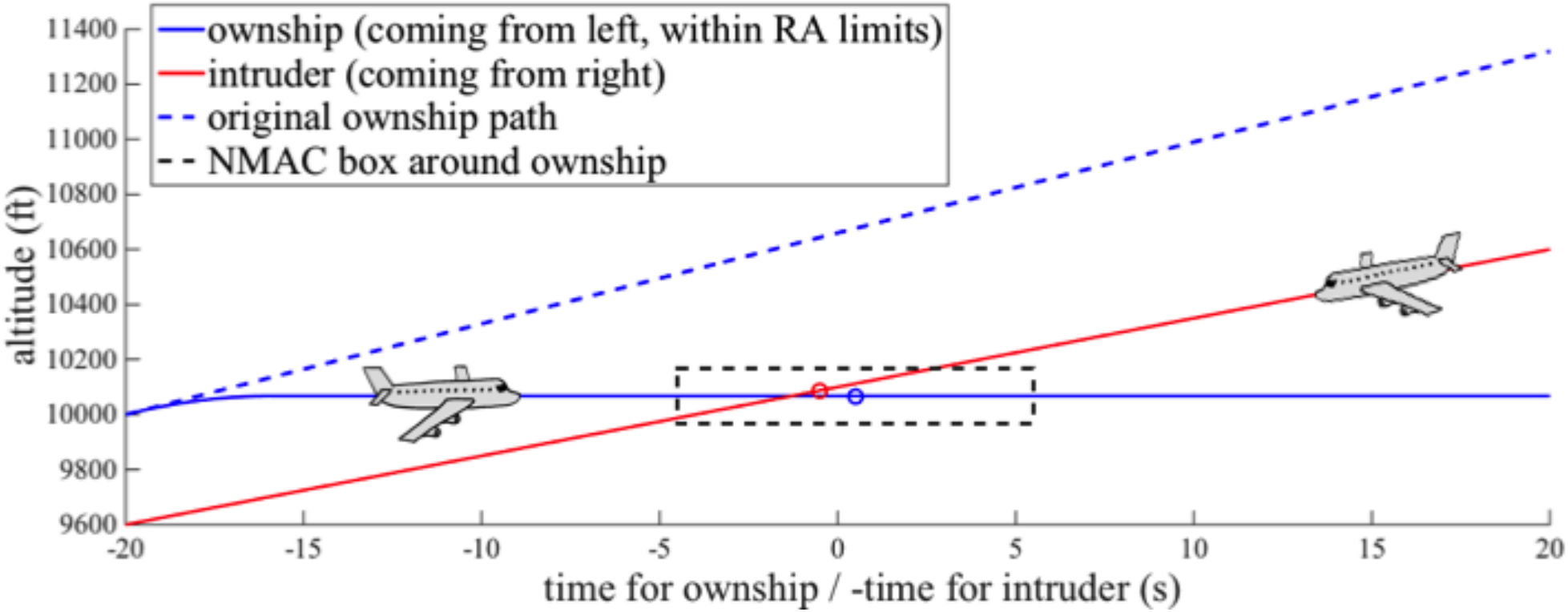


1. Identified safe region for each advisory symbolically
2. Proved safety for hybrid systems flight model in KeYmaera X

TACAS'15,EMSOFT'15,STTT'16

Airborne Collision Avoidance System ACAS X: Compare

ACAS X table comparison shows safe advisory in 97.7% of the 648,591,384,375 states compared (15,160,434,734 counterexamples).

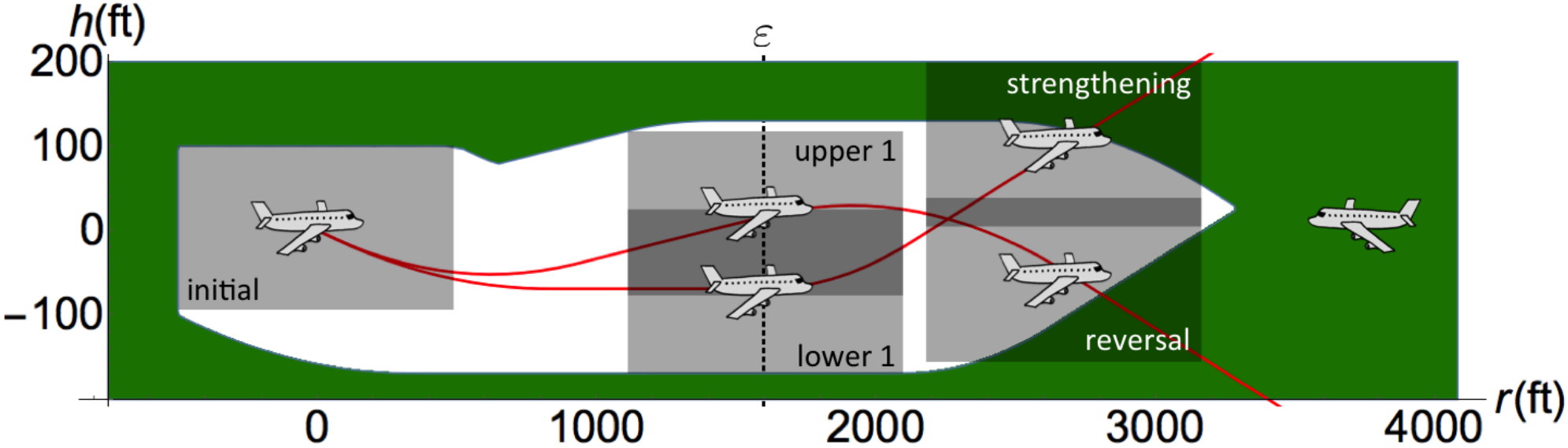


ACAS X issues DNC advisory, which induces collision unless corrected

TACAS'15,EMSOFT'15,STTT'16

Airborne Collision Avoidance System ACAS X: Refine

- Conservative, so too many counterexamples
- Settle for: safe for a little while, with safe future advisory possibility
- Safeable advisory: a subsequent advisory can safely avoid collision

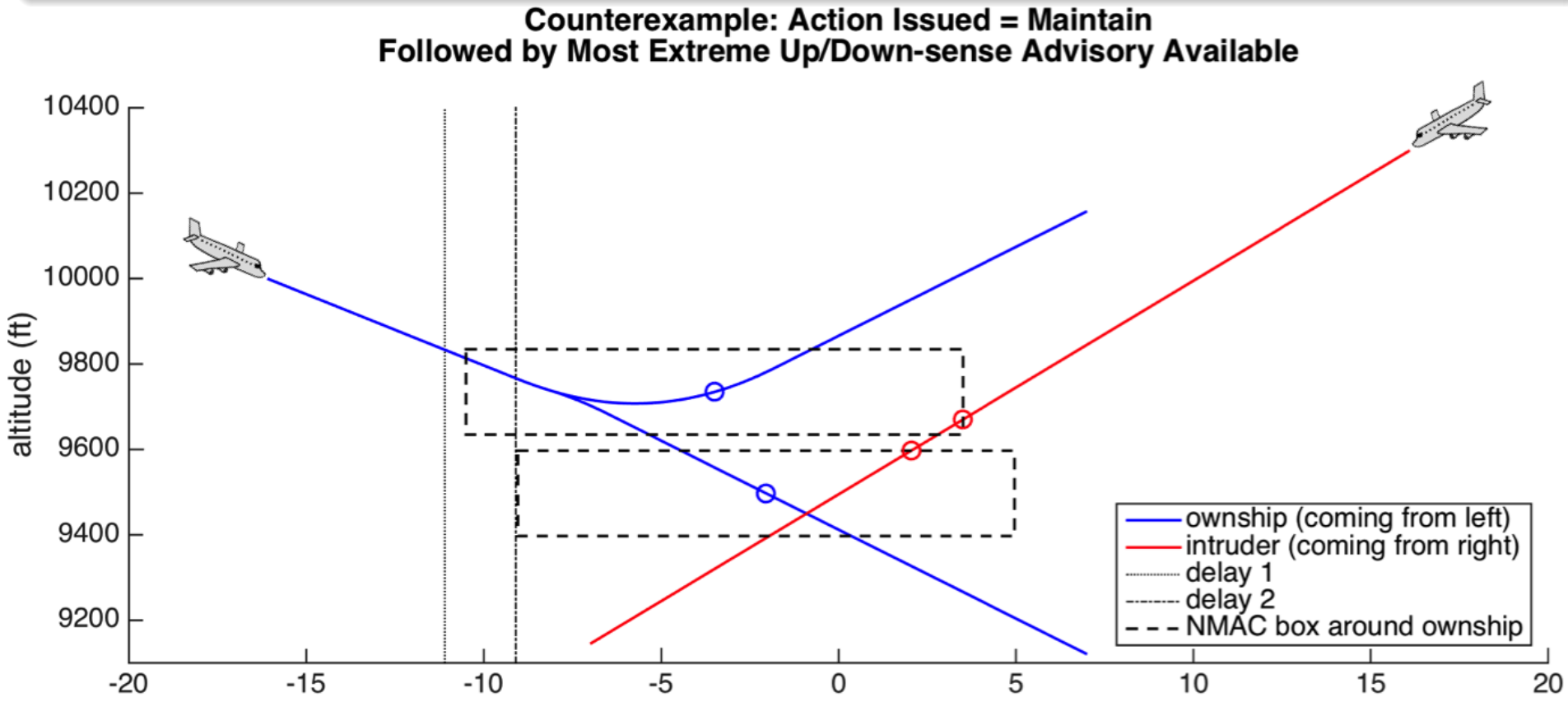


1. Identified safeable region for each advisory symbolically
2. Proved safety for hybrid systems flight model in KeYmaera X

STTT'16

Airborne Collision Avoidance System ACAS X: Compare

ACAS X table comparison shows safeable advisory in more of the 648,591,384,375 states compared (31 to 899 10^6 counterexamples).

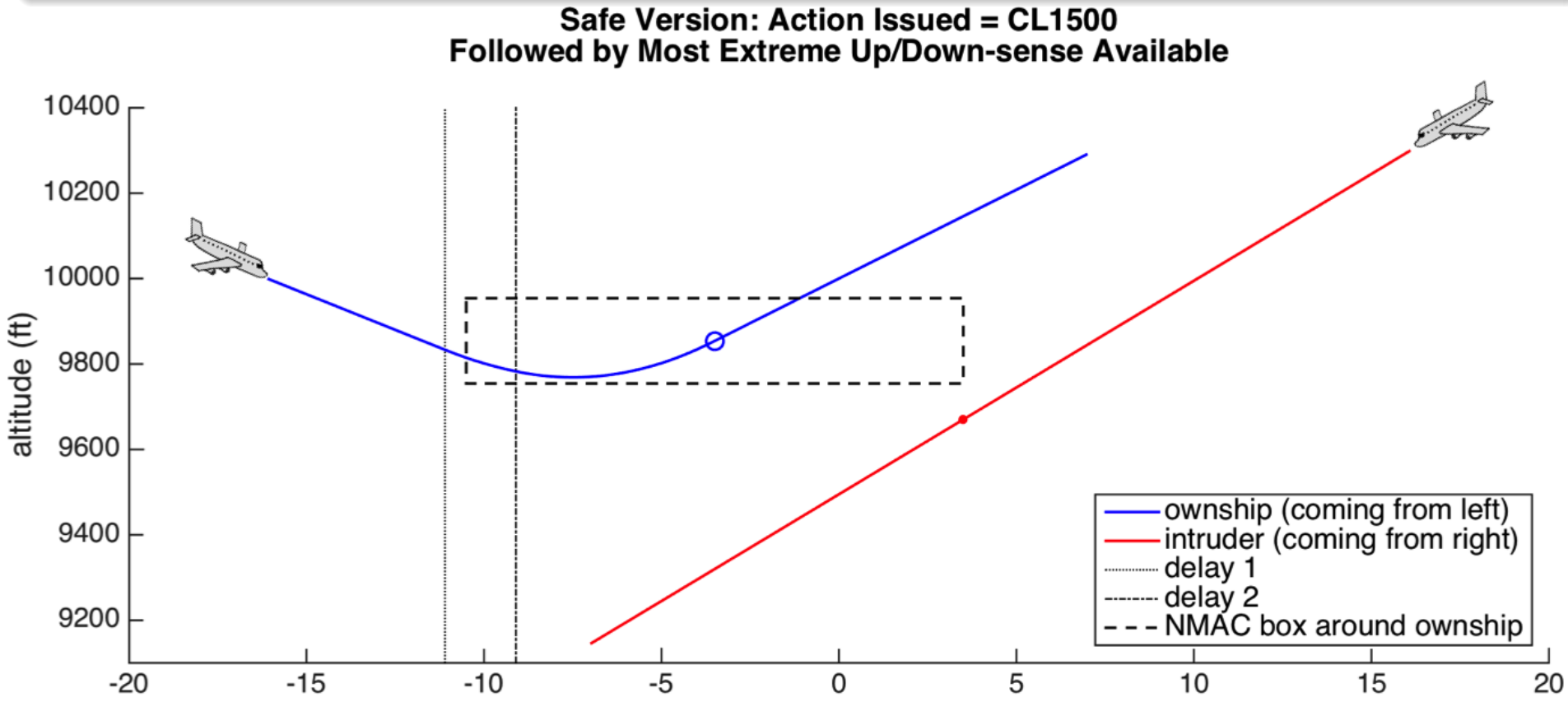


ACAS X issues Maintain advisory instead of CL1500

STTT'16

Airborne Collision Avoidance System ACAS X: Compare

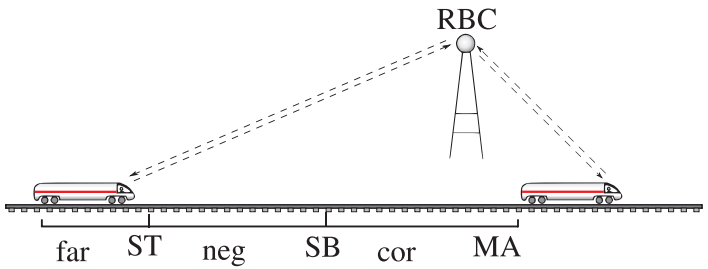
ACAS X table comparison shows safeable advisory in more of the 648,591,384,375 states compared (31 to 899 10^6 counterexamples).

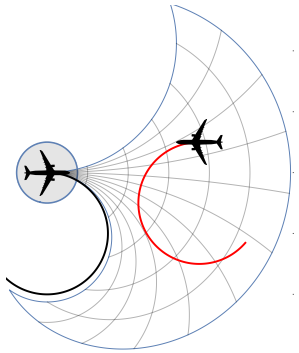


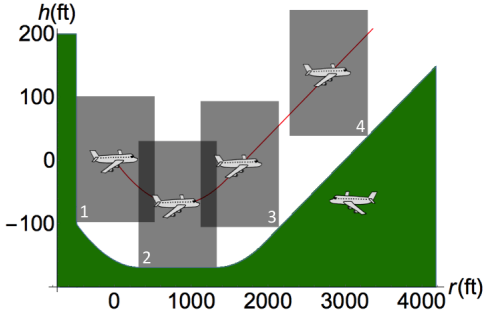
ACAS X issues Maintain advisory instead of CL1500

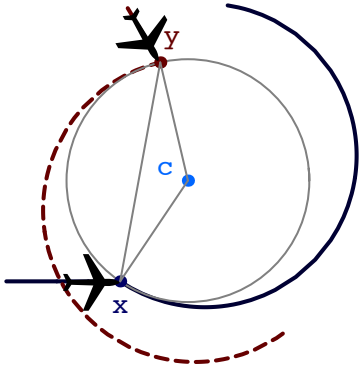
STTT'16

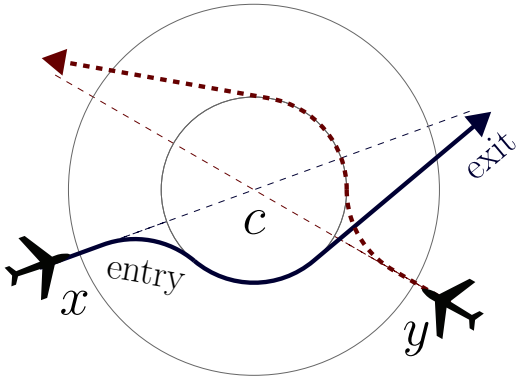
Verified CPS Applications

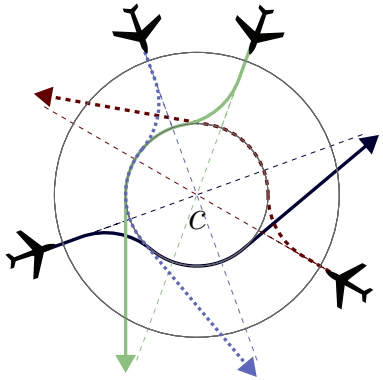


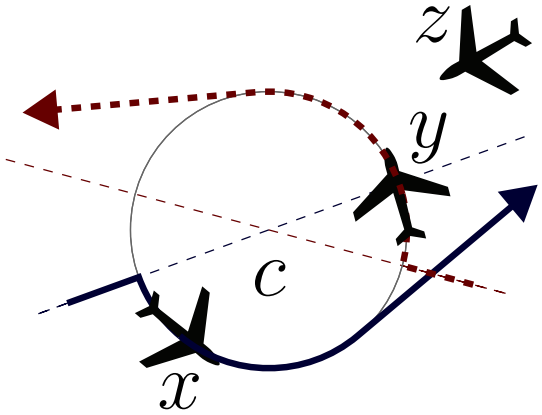


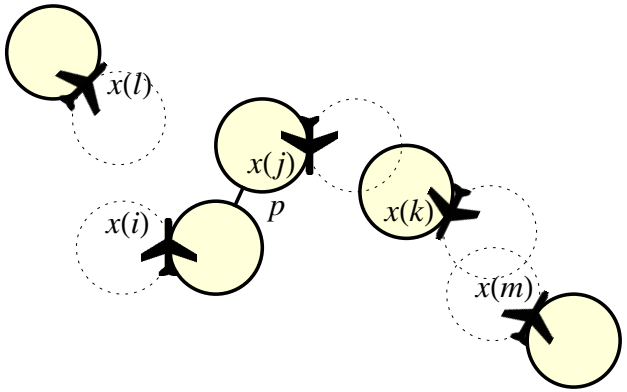


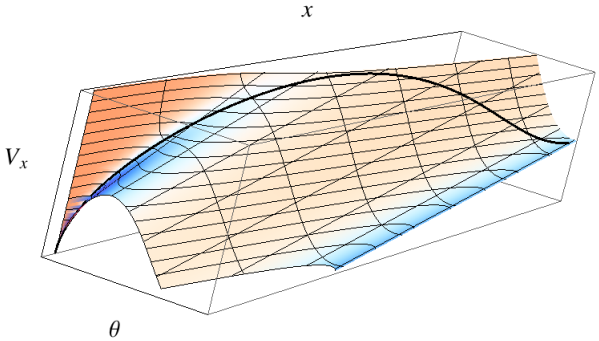






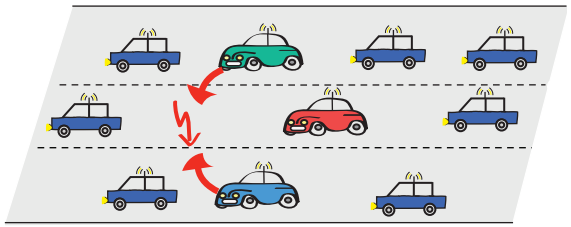


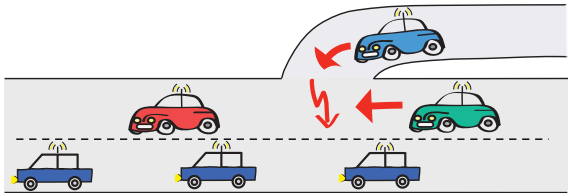


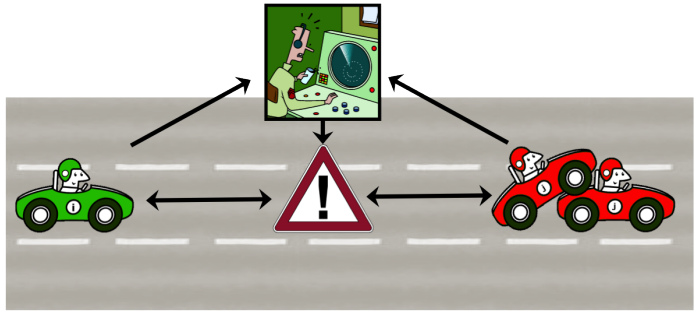


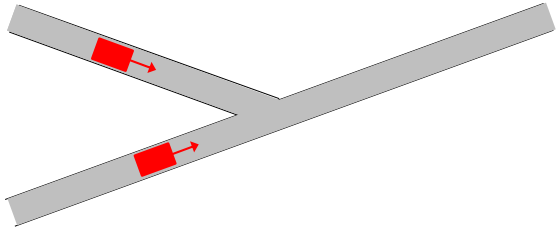
FEM'09, JAIS'14, TACAS'15, EMSOFT'15, CAV'08, FM'09, HSCC'11, HSCC'13, TACAS'14

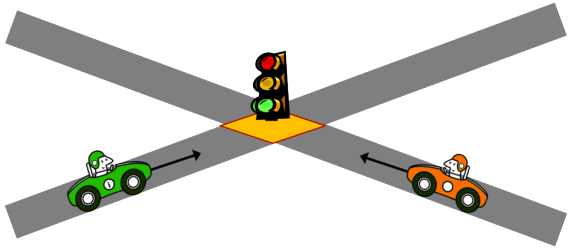
Verified CPS Applications

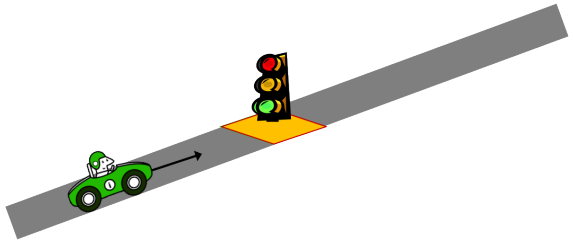


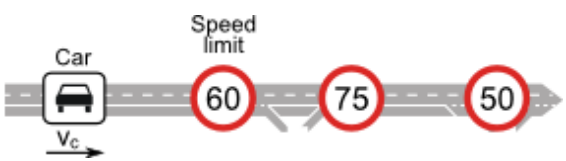


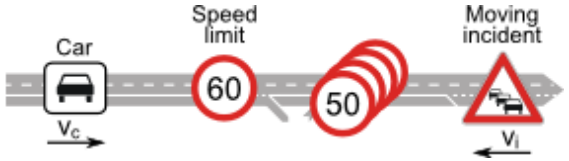


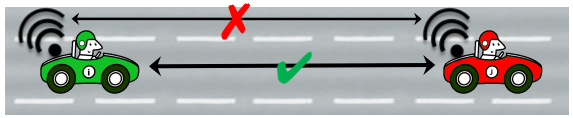


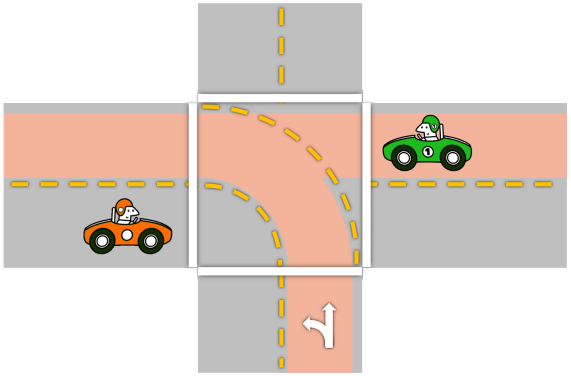


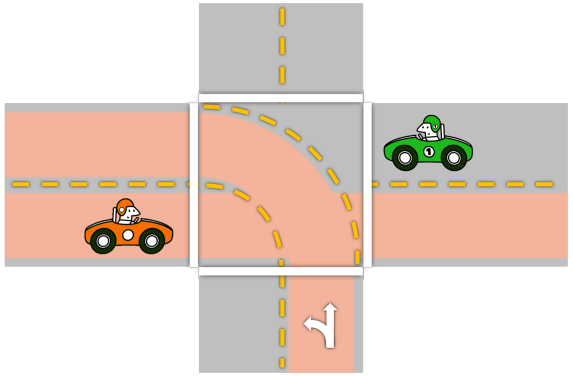


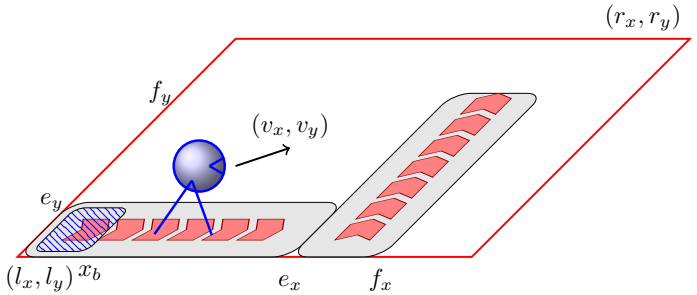






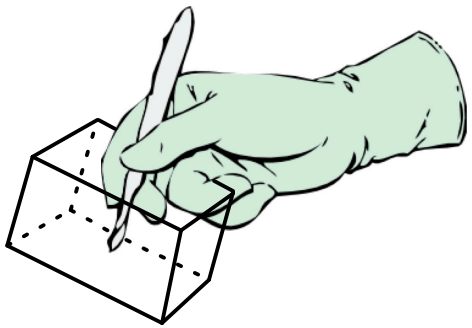


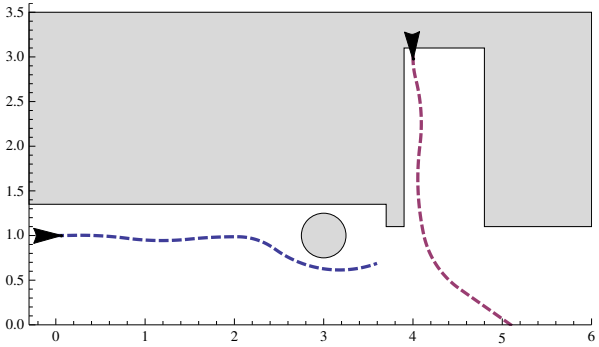


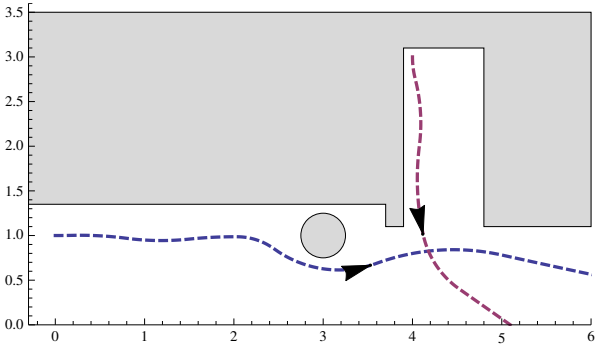


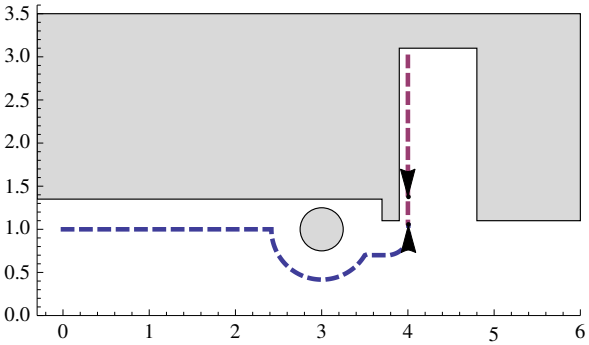
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

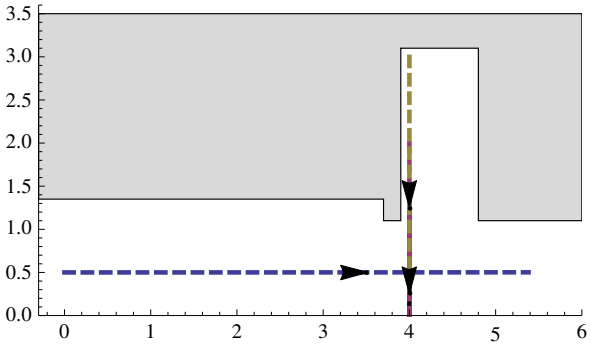
Verified CPS Applications

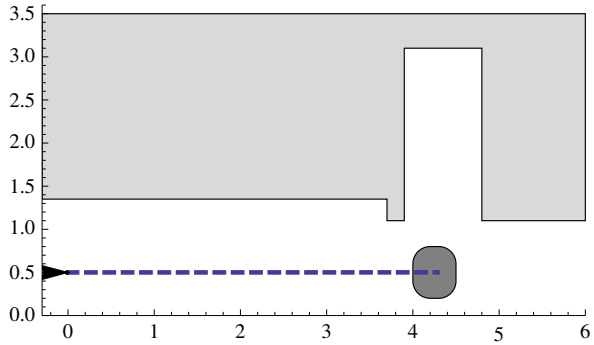


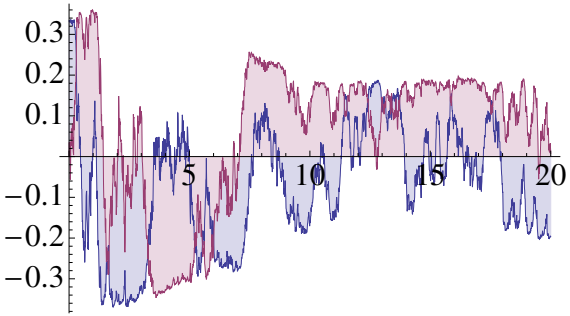


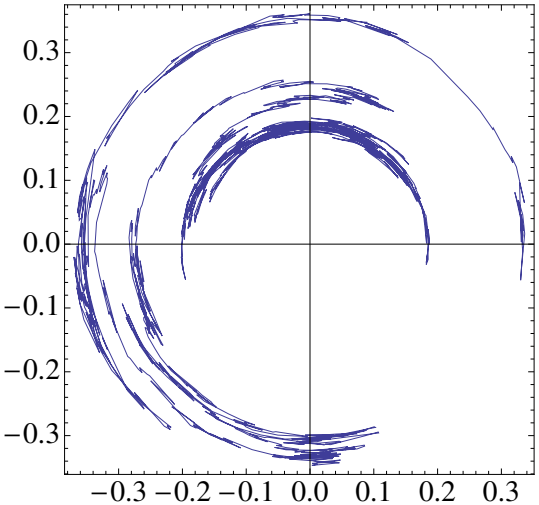


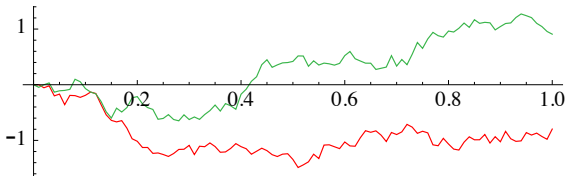




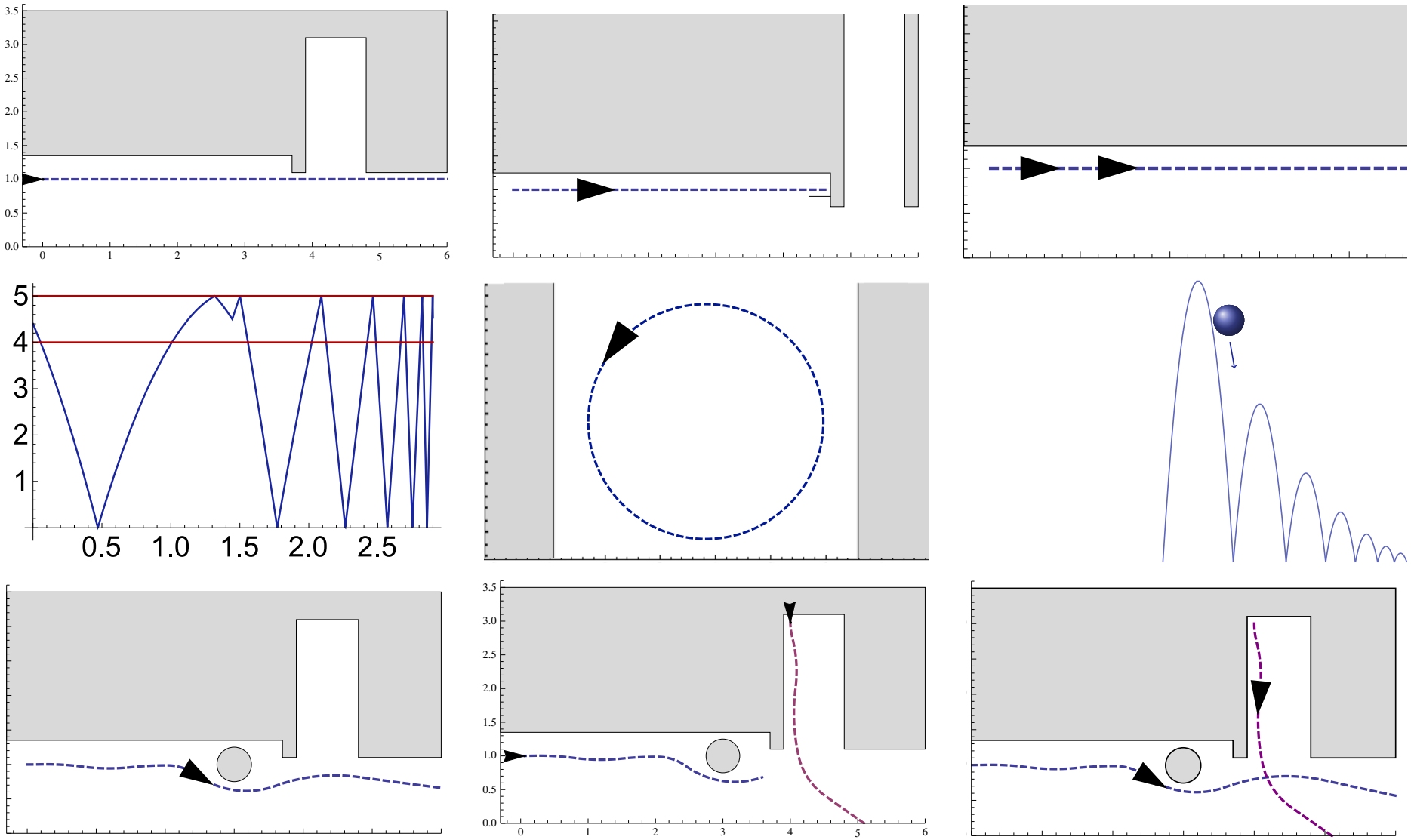






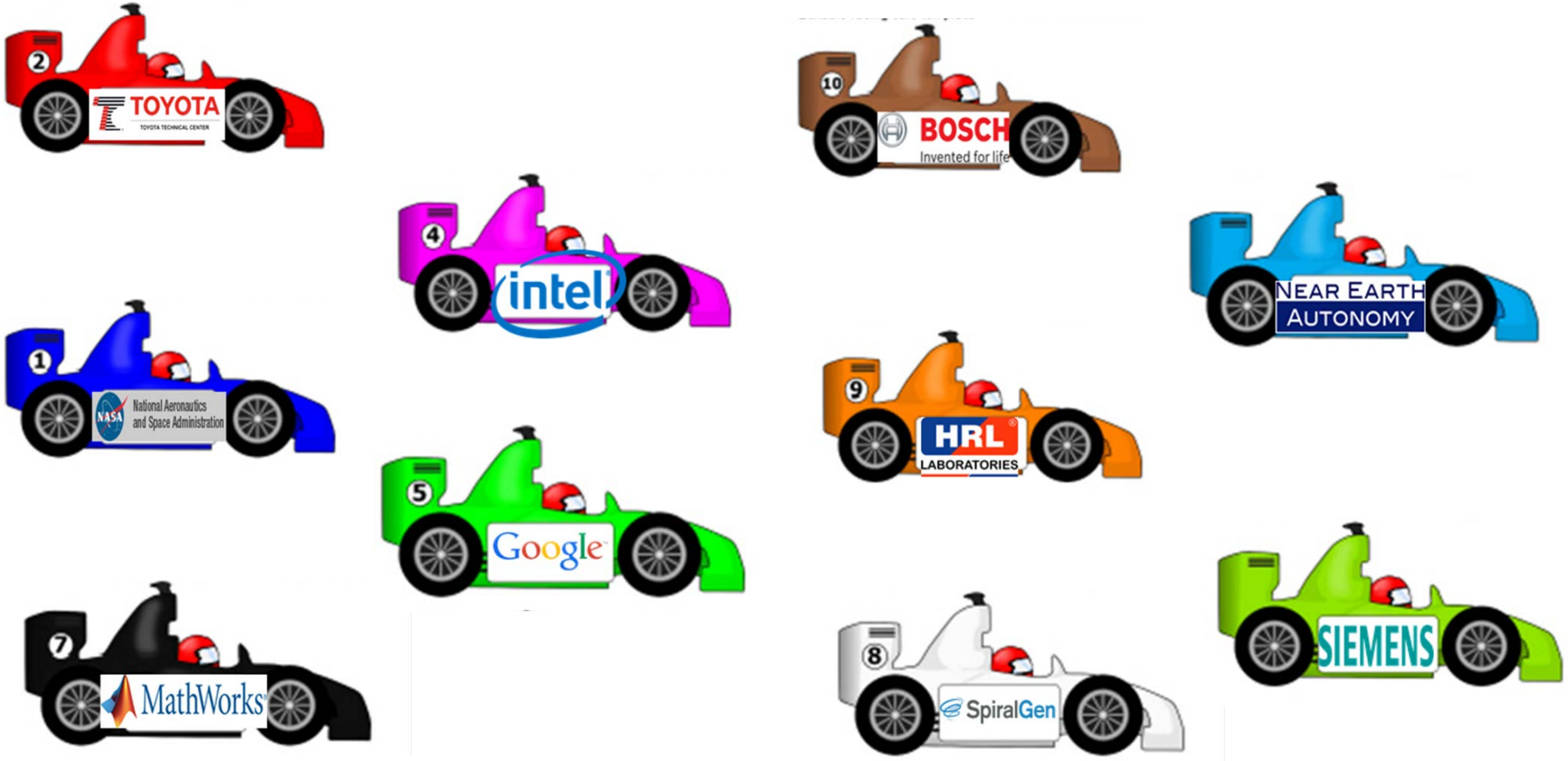


HSCC'13,RSS'13,CADE'12



15-424/624/824 *Foundations of Cyber-Physical Systems* students

Carnegie Mellon University
May 5th, 2016



An aXiomatic Tactical Theorem Prover for CPS

KeYmaera X http://keymaeraX.org/

KeYmaera X | Dashboard | Models | Proofs | Theme | Help

Escalator | Auto | Normalize | Step back

Propositional | Quantififiers | Hybrid Programs | Differential Equations | Closing | Inspect

Base case 5 | Use case 6 | Induction step 7

	-1: x>0	⊢	[?x>1;x:=x-1; ∪ {x'=v∧true}] x>0
	-2: v≥0		
loop			
	x≥2,v≥0	⊢	[{?x>1;x:=x-1; ∪ {x'=v∧true}}*] x≥0
∧L			
	x≥2∧v≥0	⊢	[{?x>1;x:=x-1; ∪ {x'=v∧true}}*] x≥0
→R			
		⊢	x≥2∧v≥0 → [{?x>1;x:=x-1; ∪ {x'=v∧true}}*] x≥0

[∪] [a∪b]P↔[a]P∧[b]P

Proof Programming

```
implyR(1) & andL(-1) & loop({`x>0`},1)
```

Run

CADE'15

An aXiomatic Tactical Theorem Prover for CPS

KeYmaera X `http://keymaeraX.org/`

Small Core	Increases trust, modularity, enables experimentation	(1652)
Tactics	Bridging between small core and powerful reasoning steps	(Hilbert) (Sequent)
Separation	Tactics can make courageous inferences Core establishes soundness	
Search&Do	Search-based tactics follow proof search strategies Constructive tactics directly build a proof	
Interaction	Interactive proofs mixed with tactical proofs and proof search	
Extensible	Flexible for new algorithms, new tactics, new logics, new proof rules, new axioms, ...	
Customize	Modular user interface, API	

CADE'15

KeYmaera X Microkernel for Soundness

	≈LOC		
KeYmaera X	1 652		hybrid prover
KeYmaera	65 989		hybrid prover
KeY	51 328		Java
Nuprl	15 000	+ 50 000	general math
MetaPRL	8 196		general math
Isabelle/Pure	8 913		general math
Coq	16 538		general math
HOL Light	396		general math
PHAVer	30 000		hybrid verifier
HSolver	20 000		hybrid verifier
SpaceEx	100 000		hybrid verifier
Flow*	25 000		hybrid verifier
dReal	50 000	+ millions	hybrid verifier
HyCreate2	6 081	+ user model analysis	hybrid verifier

Disclaimer: Self-reported estimates of the soundness-critical lines of code + rules

Uniform Substitution

Theorem (Soundness) replace all occurrences of $p(\cdot)$

$$(US)\ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of operator $\otimes$
are not free in the substitution on its argument θ (U-admissible)

$$\mathrm{US}\,\frac{[a \cup b]p(\bar{x}) \leftrightarrow [a]p(\bar{x}) \wedge [b]p(\bar{x})}{[x := x+1 \cup x' = 1]x \geq 0 \leftrightarrow [x := x+1]x \geq 0 \wedge [x' = 1]x \geq 0}$$

Acknowledgments

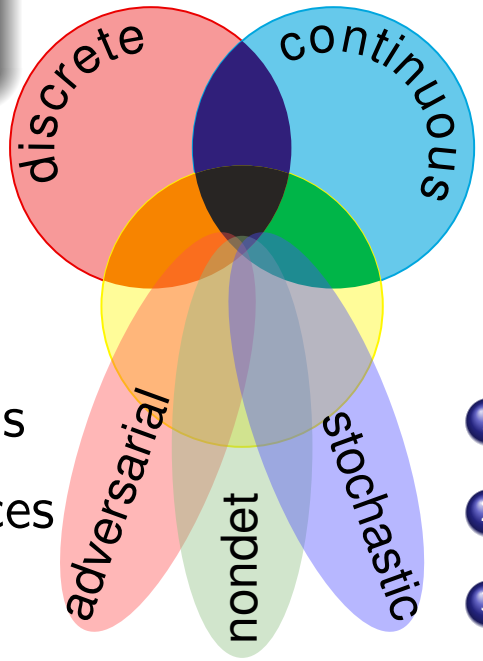
Students and postdocs of the Logical Systems Lab at Carnegie Mellon
Brandon Bohrer, Nathan Fulton, David Henriques, Sarah Loos, João Martins
Erik Zawadzki, Khalil Ghorbal, Jean-Baptiste Jeannin, Stefan Mitsch

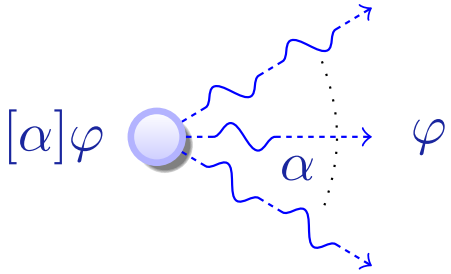
How to Prove Hybrid Systems

Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

$$\mathrm{d}\mathcal{L} = \mathrm{DL} + \mathrm{HP}$$





- Strong analytic foundations
- Practical reasoning advances
- Significant applications
- Catalyze many science areas

1. Multi-dynamical systems
2. Combine simple dynamics
3. Tame complexity
4. Complete axiomatization

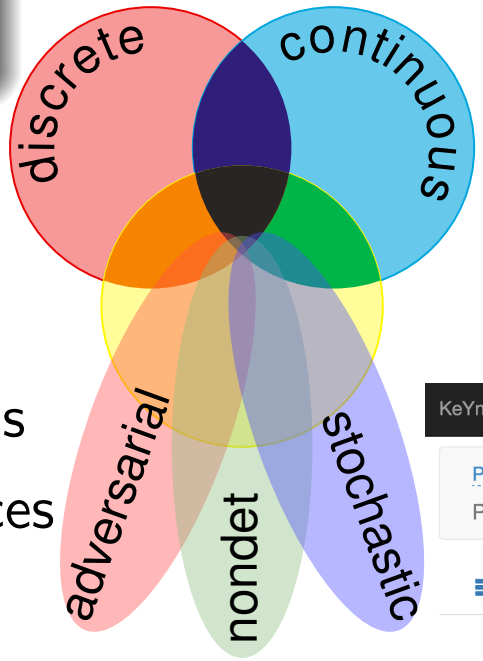
Numerous wonders remain to be discovered

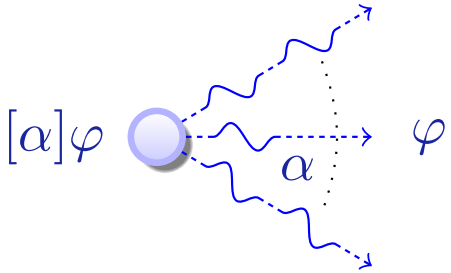
How to Prove Hybrid Systems

Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

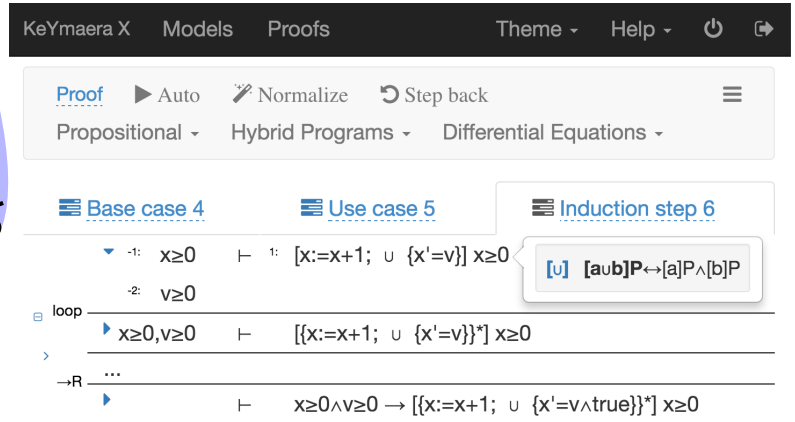
$$d\mathcal{L} = \mathsf{DL} + \mathsf{HP}$$



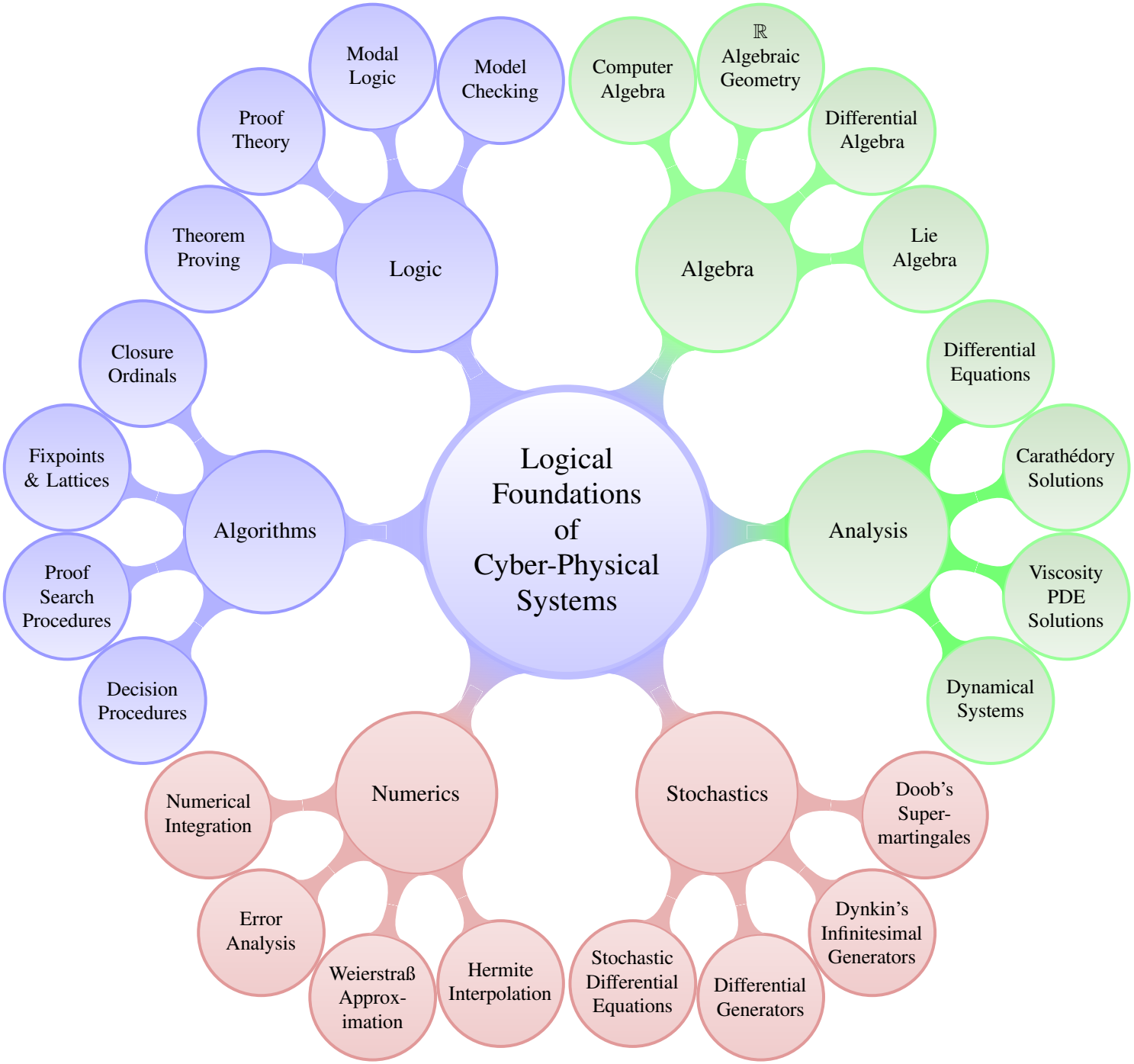


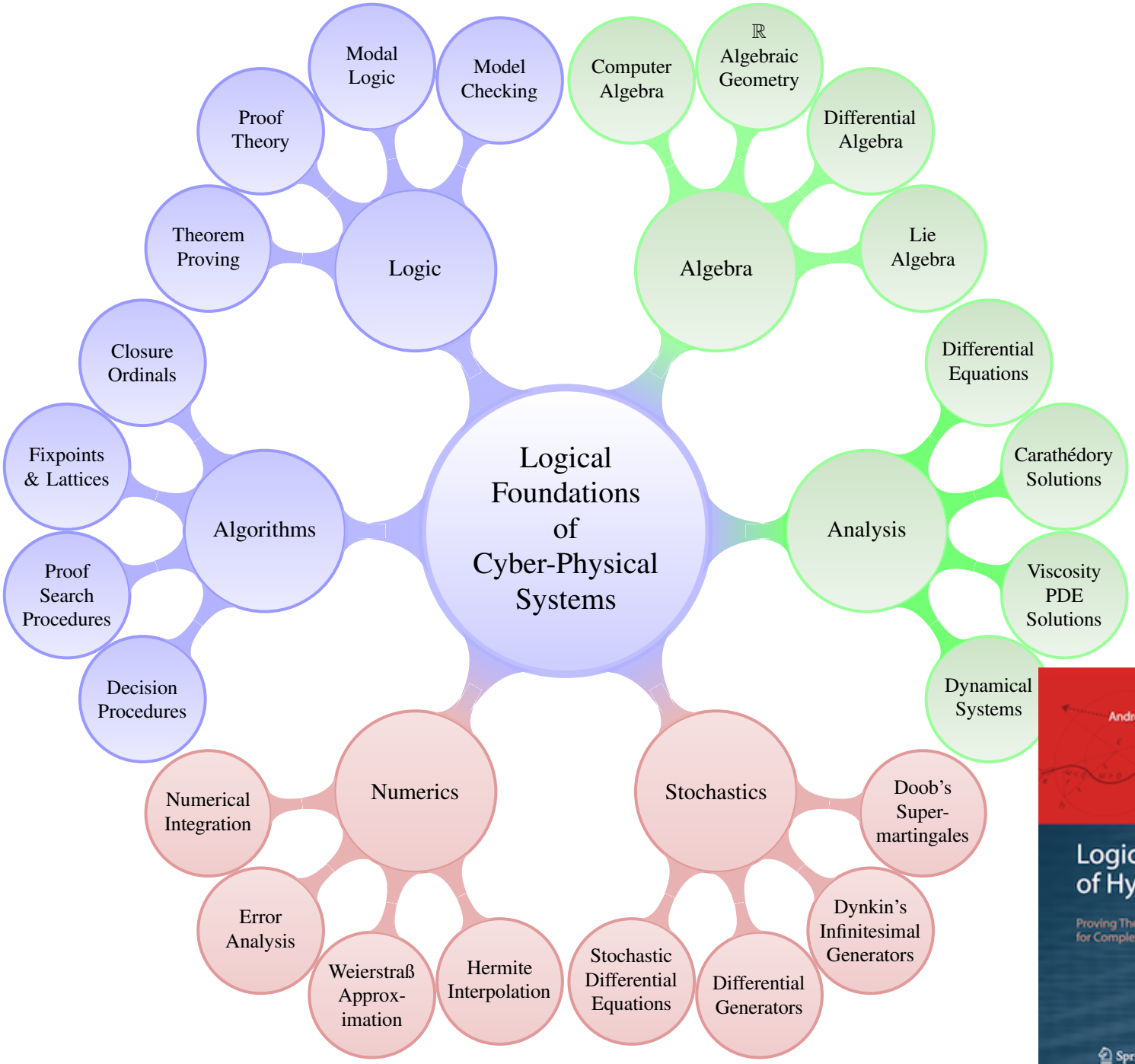
- Strong analytic foundations
- Practical reasoning advances
- Significant applications
- Catalyze many science areas

KeYmaera X



Numerous wonders remain to be discovered


Logical Foundations of Cyber-Physical Systems
Logic
Modal Logic
Model Checking
Proof Theory
Theorem Proving
Algebra
Computer Algebra
ℝ Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Carathédory Solutions
Viscosity PDE Solutions
Dynamical Systems
Stochastics
Doob's Super-martingales
Dynkin's Infinitesimal Generators
Differential Generators
Stochastic Differential Equations
Numerics
Hermite Interpolation
Weierstraß Approx-imation
Error Analysis
Numerical Integration
Algorithms
Closure Ordinals
Fixpoints & Lattices
Proof Search Procedures
Decision Procedures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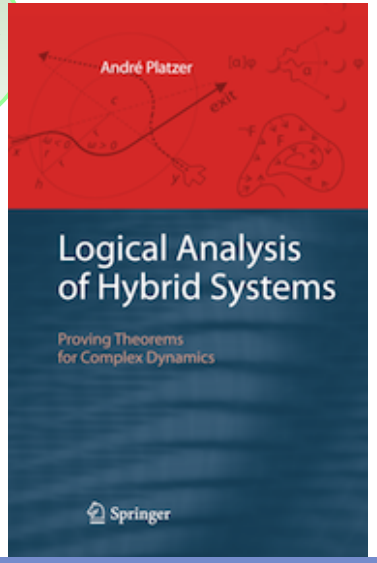
André Platzer
Logical Analysis of Hybrid Systems
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Springer

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program semantics) $(\llbracket\cdot\rrbracket : \text{HP} \to \wp(\mathcal{S} \times \mathcal{S}))$

$$
\begin{aligned}
\llbracket x := e \rrbracket &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \llbracket x \rrbracket \nu = \llbracket e \rrbracket \omega\} \\
\llbracket ?Q \rrbracket &= \{(\omega, \omega) \ : \ \omega \in \llbracket Q \rrbracket\} \\
\llbracket x' = f(x) \rrbracket &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
\llbracket \alpha \cup \beta \rrbracket &= \llbracket \alpha \rrbracket \cup \llbracket \beta \rrbracket \\
\llbracket \alpha ; \beta \rrbracket &= \llbracket \alpha \rrbracket \circ \llbracket \beta \rrbracket \\
\llbracket \alpha^* \rrbracket &= \bigcup_{n \in \mathbb{N}} \llbracket \alpha^n \rrbracket
\end{aligned}
$$

Definition (d$\mathcal{L}$ semantics) $(\llbracket\cdot\rrbracket : \text{Fml} \to \wp(\mathcal{S}))$

$$
\begin{aligned}
\llbracket e \geq \tilde{e} \rrbracket &= \{\omega \ : \ \llbracket e \rrbracket \omega \geq \llbracket \tilde{e} \rrbracket \omega\} \\
\llbracket \neg P \rrbracket &= \llbracket P \rrbracket^{\complement} \\
\llbracket P \wedge Q \rrbracket &= \llbracket P \rrbracket \cap \llbracket Q \rrbracket \\
\llbracket \langle \alpha \rangle P \rrbracket &= \llbracket \alpha \rrbracket \circ \llbracket P \rrbracket = \{\omega \ : \ \nu \in \llbracket P \rrbracket \text{ for some } \nu : \ (\omega, \nu) \in \llbracket \alpha \rrbracket\} \\
\llbracket [\alpha] P \rrbracket &= \llbracket \neg \langle \alpha \rangle \neg P \rrbracket = \{\omega \ : \ \nu \in \llbracket P \rrbracket \text{ for all } \nu : \ (\omega, \nu) \in \llbracket \alpha \rrbracket\} \\
\llbracket \exists x \, P \rrbracket &= \{\omega \ : \ \omega_x^r \in \llbracket P \rrbracket \text{ for some } r \in \mathbb{R}\}
\end{aligned}
$$

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